TODA LATTICE AND TOMEI'S THEOREM

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ABSTRACT. The following notes are for the talk given by the authors at Northeastern Research Seminar in Mathematics in Fall 2018. After introducing the Hamiltonian systems formalism and Toda systems, we provide an overview of C. Tomei's results on the topology of isospectral varieties.

CONTENTS

1. GENERAL FACTS ON TODA SYSTEMS

We start by recalling the basic definitions.

Definition 1.1. By a *Hamiltonian system* in the phase space $\mathbb{R}^{2n}(\mathbf{p}, \mathbf{q})$ we will understand a system

$$\begin{cases} \dot{p}_i = -\frac{\partial H}{\partial q_i} \\ \dot{q}_i = \frac{\partial H}{\partial p_i}, \end{cases} \quad i \in \{1 \ldots, n\} \tag{1}$$

where the function $H(\mathbf{p}, \mathbf{q})$ is called the *Hamiltonian* of the system. Hereafter we use $\cdot$ to denote the derivative with respect to time.

Definition 1.2. Given two functions $F(\mathbf{p}, \mathbf{q}, t)$ and $G(\mathbf{p}, \mathbf{q}, t)$, their *Poisson bracket* $\{F, G\}$ is a function, s.t. for any three functions F, G, H the following identities hold:

(1) $\{F, G\} = -\{G, F\}$ (anti-commutativity);
(2) $\{aF + bH, G\} = a\{F, G\} + b\{F, H\}$ for any $a, b \in \mathbb{R}$ (bilinearity);
(3) $\{FG, H\} = F\{G, H\} + \{F, H\}G$ (Leibnitz rule);
(4) $\{\{F, G\}, H\} + \{\{H, F\}, G\} + \{\{G, H\}, F\} = 0$ (Jacobi identity).

Remark 1.3. There exist canonical coordinates (also known as Darboux coordinates) $(\mathbf{p}, \mathbf{q})$ on the phase space, s.t. the Poisson bracket takes the form

$$\{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i},$$

we will work in this coordinates.

Definition 1.4. A function $F = F(\mathbf{p}, \mathbf{q}, t) : \mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R}$, which satisfies $\dot{F} = 0$ and $\{H, F\} = 0$ is called a *first integral of motion*. Such functions form an algebra under the Poisson bracket. A Hamiltonian system is called *integrable* if there are n independent integrals of motion $F_1, \dots, F_n$, which pairwise Poisson commute, i.e. $\{F_i, F_j\} = 0\ \forall i, j \in \{1, \dots, n\}$.

Example 1.5. Since $\{H, H\} = 0$, the Hamiltonian H itself is a first integral of motion.

Consider the Hamiltonian system on $\mathbb{R}^{2n}(\mathbf{p}, \mathbf{q})$ with Hamiltonian $H(\mathbf{p}, \mathbf{q}) := \frac{1}{2}\sum\limits_{i=1}^{n} p_i^2 + \sum\limits_{j=1}^{n-1} e^{q_j - q_{j+1}}$. this is the phase space of n ordered points on a line, where each point interacts only with its immediate neighbors with respect to the order.

$$\begin{cases} \dot{p}_i = e^{q_{i-1}-q_i} - e^{q_i - q_{i+1}},\ j \in \{1, \dots, n\} \\ \dot{q}_i = p_i \end{cases} \tag{2}$$

The variables

$$\begin{cases} a_j = -\frac{1}{2}p_j,\ j \in \{1, \dots, n\} \\ b_k = \frac{1}{2}e^{\frac{1}{2}(q_k - qk+1)},\ k \in \{1, \dots, n-1\} \end{cases} \tag{3}$$

are called *Flashke variables*. We also set $b_0 = b_n = 0$. In these variables the equation transforms into

$$\begin{cases} \dot{a}_j = 2(b_j^2 - b_{j-1}^2),\ j \in \{1, \dots, n\} \\ \dot{b}_k = b_k(a_{k+1} - a_k),\ k \in \{1, \dots, n-1\} \end{cases} \tag{4}$$

It is direct to check that the system of equations (4) is equivalent to the Lax equation

$$\dot{L} = [B(L), L] \tag{5}$$

with

$$L = \begin{pmatrix} a_1 & b_1 & 0 & \dots & 0 \\ b_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & b_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a_{n-1} & b_{n-1} \\ 0 & \dots & 0 & b_{n-1} & a_n \end{pmatrix}, B(L) = \begin{pmatrix} 0 & b_1 & 0 & \dots & 0 \\ -b_1 & 0 & b_2 & \ddots & \vdots \\ 0 & -b_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & b_{n-1} \\ 0 & \dots & 0 & -b_{n-1} & 0 \end{pmatrix}$$

Theorem 1.6. *The functions* $F_k := \frac{1}{k}\mathrm{tr}(L^k),\ k \in \{1, \dots, n\}$ *are first integrals of motion. They are independent and pairwise Poisson commute. Therefore the Toda system is integrable.*

2. Symplectic form on Jacobi varieties

The goal of this section is to explain the appearance of symplectic form on certain class of varieties, called Jacobi varieties. We begin with definitions.

Define the subset $\mathcal{T}_n$ of real-valued three-diagonal symmetric matrices $n \times n$-matrices:

$$L = \begin{pmatrix} a_1 & b_1 & 0 & \dots & 0 \\ b_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & b_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a_{n-1} & b_{n-1} \\ 0 & \dots & 0 & b_{n-1} & a_n \end{pmatrix}$$

Following [7] we first introduce the set of Jacobian matrices $J_n \subset \mathcal{T}_n$ as the subset formed by matrices with positive off-diagonal elements

More generally, let (X, ω) be a *symplectic variety*, i.e. a variety equipped with a closed, nondegenerate, 2-form ω. Any Hamiltonian $H : X \to \mathbb{R}$ gives rise to a vector field X_H via the equation

$$\omega_x(X_H(x), \bullet) = dH_x(\bullet),$$

where $x \in X$ is any point and dH is the 1- form produced by the differential of H.

A Hamiltonian vector field X_H is said to be completely integrable provided there exist n commuting Hamiltonians $H_k : X \to \mathbb{R}$ (i.e., their induced vector fields $X_k := X_{H_k}$ pairwise commute and commute with X_H, or equivalently, $\{H_i, H\} = \{H_i, H_j\} = 0$ for the Poisson bracket induced by the symplectic form. Finally, the Hamiltonians H_k should be *functionally independent* (their gradients should be linearly independent on a dense subset of X).

Let U^+ be the Lie group of real-valued upper-triangular matrices with positive values on the diagonal and $\mathfrak{u} := \mathrm{Lie}(U^+)$ be the corresponding Lie algebra. It consists of real-valued upper-triangular matrices. The dual of $\mathfrak{u}$ can be identified with real-valued symmetric matrices. We will denote it by $\mathcal{S}$. Recall that there is an adjoint action of U^+ on its Lie algebra $\mathfrak{u}$ given by

$$Ad_g(u) := gug^{-1}$$

and dually, the coadjoint action of U^+ on $\mathcal{S}$ via

$$Ad^*_g(\alpha)(u) := \langle Ad^*_g(\alpha), u\rangle = \langle \alpha, g^{-1}ug\rangle = tr(\alpha g^{-1}ug) = tr(g\alpha g^{-1}u),$$

where $g \in U^+, u \in \mathfrak{u}$ and $\alpha \in \mathcal{S}$. Here we run into a problem, since the matrix $g\alpha g^{-1}$ does not have to be symmetric, i.e. might not be an element of $\mathcal{S}$. However, there is an easy fix. Namely any matrix A can be written as a sum of symmetric and a strictly upper triangular matrices:

$$A = \Pi_S A + \Pi_{uT} A$$

in a unique way and, as trace is an additive function, it equals to the trace of the symmetric part. This enables us to define $Ad^*_g(\alpha) := \Pi_S(g\alpha g^{-1})$.

Now let $\alpha \in \mathcal{S}$ be a matrix and $\mathcal{O}_\alpha$ its coadjoint orbit. It turns out that $\mathcal{O}_\alpha$ admits a canonically defined symplectic form and, therefore, is a symplectic manifold (we refer the reader to Proposition 1.1.5 in [1] for this construction).

It is direct to compute that the initial Hamiltonian $H(p, q)$ for the Toda lattice converts to $H(S) = tr\frac{L^2}{2}$ in Flaschka's variables.

We conclude this section with an important remark.

Remark 2.1. Jacobi matrices with fixed trace form a coadjoint orbit (see [4]) , therefore, such varieties are symplectic.

$$\begin{tikzcd}
 & (\lambda_1,\lambda_2,\lambda_3) & \\
(\lambda_1,\lambda_3,\lambda_2) \arrow[ur] & & (\lambda_2,\lambda_1,\lambda_3) \arrow[ul] \\
(\lambda_2,\lambda_3,\lambda_1) \arrow[u] & & (\lambda_3,\lambda_1,\lambda_2) \arrow[u] \\
 & (\lambda_3,\lambda_2,\lambda_1) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

FIGURE 1. Permutohedron $\mathcal{P}_\Lambda$ for $\bar{\mathcal{J}}_\Lambda$ and $n = 3$

3. APPLICATION: TOMEI'S THEOREMS

In 1984 Tomei in [6] (see also [7] for a more recent overview of the basic results and constructions) obtained important results on the topology of the variety $\mathcal{T}_n$. By an *isospectral* subvariety $X \subset J_n \subset \mathcal{T}_n$ we mean that the matrices in X have the same eigenvalues. Henceforth we assume that the eigenvalues $\lambda_1 > \lambda_2 > \ldots > \lambda_n$ are distinct and denote the corresponding isospectral variety by J_Λ, analogously $\mathcal{T}_\Lambda$, where $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. Tomei has shown that J_Λ is diffeomorphic to $\mathbb{R}^{n-1}$. For a better understanding of $\mathcal{T}_\Lambda$ it is helpful to study the closure $\overline{J_\Lambda}$ of J_Λ inside the space of real-valued symmetric matrices.

We will need one more definition.

Definition 3.1. The *permutohedron* $\mathcal{P}(x_1, \ldots, x_n)$ is the convex polytope in $\mathbb{R}^n$ defined as the convex hull of all permutations of the vector $(x_1, \ldots, x_n)$.

The main result on the topology of $\bar{J}_\Lambda \to \mathcal{P}(\lambda_1, \ldots, \lambda_n)$ is the following theorem (see Theorem 3.1 in [7] and references therein).

Theorem 3.2. *The map $\bar{J}_\Lambda \to \mathcal{P}(\lambda_1, \ldots, \lambda_n)$ sending $Q^T \Lambda Q$ to diag$(Q\Lambda Q^T)$ is a homeomorphism.*

Example 3.3. Consider a variety $\bar{J}_\Lambda$ for $n = 3$. The corresponding permutohedron $\mathcal{P}_\Lambda$ is depicted on Figure 3. The vertices correspond to diagonal matrices with spectrum Λ and the edges are naturally indexed by transpositions. They correspond to block diagonal matrices with a symmetric 2×2 block, whose location is determined by the elements exchanged by the transposition. For instance, the matrices on the edge joining $(\lambda_1, \lambda_2, \lambda_3)$ and $(\lambda_1, \lambda_3, \lambda_2)$ are of the form

$$L = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & a_2 & b_2 \\ 0 & b_2 & a_3 \end{pmatrix}.$$

The direction of the arrows reflects the flow of the Hamiltonian vector field given by 5.

In order to make the transition from $\bar{J}_\Lambda$ to the full isospectral manifold $\mathcal{T}_\Lambda$, one needs the construction of an appropriate Coxeter group (see [6]). A quick overview: dropping the signs of the off-diagonal entries of a real-valued, symmetric tridiagonal matrix (preserving its symmetry) does not change the eigenvalues of a matrix in $\mathcal{T}_\Lambda$. Therefore, it is sufficient to work with Jacobi matrices. In particular, the sets of matrices in $\mathcal{T}_\Lambda$ with nonzero entries splits into 2^{n-1} connected components, each isomorphic to J_Λ. To get $\mathcal{T}_\Lambda$ it remains to take the closure of these components and glue them along faces which are naturally identified.

The variety $\mathcal{T}_\Lambda$ is a compact manifold of dimension $n-1$. However, if the eigenvalues are not distinct, $\mathcal{T}_\Lambda$ is not a manifold. For example, $\mathcal{T}_{\lambda,\lambda,\mu}$ is homeomorphic to $S^1 \vee S^1$, the bouquet of two circles.

Remark 3.4. The compactness of $\mathcal{T}_\Lambda$ can be established using that it is a closed submanifold of the orbit $\mathcal{O}_\Lambda$, which is compact.

Exercise 3.5. Construct the homeomorphism $\mathcal{T}_{\lambda,\lambda,\mu} \to S^1 \vee S^1$.

Remark 3.6. The variety $\mathcal{T}_\Lambda$ is the level surface for $F_k = \frac{1}{k}\sum_{i=1}^{n} \lambda_i^k$.

The following theorem is the main result of [6].

Theorem 3.7. *The Euler characteristic of $\mathcal{T}_\Lambda$, is $\chi(\mathcal{T}_\Lambda) = (-1)^{n+1}\frac{2^{n+1}(2^{n+1}-1)}{n+1}B_{n+1}$, where B_n, is the nth Bernoulli number.*

Proof. We present the scheme of the proof. For a more detailed version the reader is referred to [6].

Step 1 The system $\dot{L} = [B(L), L]$ generates a globally defined flow φ_t on J_n, leaving each isospectral subvariety invariant.

Step 2 The vector field $\dot{L} = [B(L), L]$ has $n!$ fixed points on $\mathcal{T}_\Lambda$. Denote the set of equilibrium points by $\mathcal{T}_\Lambda^\nu$. These points are diagonal matrices $\mathrm{diag}(\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \ldots, \lambda_{\sigma(n)})$, i.e. indexed by permutations $\sigma \in S_n$. Indeed, recall that in variables $(a_j, b_k), k, j \in \{1, \ldots, n\}$ the equation $\dot{L} = [B(L), L]$ reads as

$$\begin{cases} \dot{a}_j = 2(b_j^2 - b_{j-1}^2), \ j \in \{1, \ldots, n\} \\ \dot{b}_k = b_k(a_{k+1} - a_k), \ k \in \{1, \ldots, n-1\} \end{cases} \tag{6}$$

This immediately implies that for a fixed point p all b_j coordinates must be equal and since $b_0 = 0$ all equal to 0. So, p must be a diagonal matrix.

Step 3 The index of the vector field at a fixed point $p \in \mathcal{T}_\Lambda^\nu$, corresponding to a permutation σ is equal to $(-1)^r(\sigma)$, where $r(p)$ is the number of pairs $(i, i+1)$ such that $\sigma(i) < \sigma(i+1)$ for the permutation σ corresponding to p. The number of equilibrium points with $r(p) = k$ is equal to $E(n,k) = \left\langle {n \atop k+1} \right\rangle$, where $E(n,k)$ are called the *Eulerian numbers*.

Step 4 The Poincare-Hopf index theorem asserts that $\chi(\mathcal{T}_\Lambda) = \sum_{x \in \mathcal{T}_\Lambda^\nu} \mathrm{ind}_x(\nu) = \sum_{k=0}^{n} (-1)^k E(n,k) = \sum_{k=0}^{n} (-1)^k \left\langle {n \atop k+1} \right\rangle = (-1)^{n+1}\frac{2^{n+1}(2^{n+1}-1)}{n+1}B_{n+1}$, where the last equality can be found as Exercise 3 after Section 1.3, Chapter 5 in [3].

□

Remark 3.8. Tomei has shown that $\mathcal{T}_\Lambda$ is an orientable manifold and its universal cover is $\mathbb{R}^{n-1}$ (see Sections 4, 5). In [2] Fried proved that the Betti number $b_k(M)$ is equal to $\left\langle {n \atop k+1} \right\rangle$. He also described the multiplication in the cohomology ring.

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