Geometric Satake equivalence

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Langlands dual groups

Connected complex reductive groups come in pairs of Langlands dual groups G, $\check{G}$.

G	$\check{G}$
$GL(n)$	$GL(n)$
$SO(2n)$	$SO(2n)$
$SL(n)$	$PGL(n)$
$SO(2n+1)$	$Sp(2n)$

- The Dynkin diagram of $Lie(\check{G})$ is obtained from $Lie(G)$ by reversing the arrows.
- We have $\pi_1(G) \approx Z(\check{G})$.
- If $(X^*, \Phi, X_*, \Phi^\vee)$ is a root datum for G, then $(X_*, \Phi^\vee, X_*, \Phi)$ is a root datum for $\check{G}$.

- Fix a connected reductive group G over $\mathbb{C}$. Can we define $\check{G}$ without root data?
- Denote by Gr_G the affine Grassmannian of G.
- Recall that $G_O = J(G) = G(\mathbb{C}[[t]])$ acts on Gr_G.

Theorem (Drinfeld, Lusztig, Ginzburg, Mirkovic, Vilonen)

We have an equivalence of categories:

$$Perv^{G_O}(Gr_G) \simeq Rep(\check{G}).$$

Here $Perv^{G_O}$ is the category of G_O-equivariant perverse sheaves, Rep is the category of finite dimensional complex representations.

Crash-course on perverse sheaves

- Work with sheaves of complex vector spaces
- Local systems on a topological space = locally constant sheaves = representations of π_1.
- Constructible sheaves: sheaves such that there is a stratification of the topological space such that the restrictions to the strata are local systems.
- $D^b(X)$ is the derived category of constructible sheaves on X.
- There is an abelian subcategory $Perv(X)$ inside $D^b(X)$ different from the abelian category of constructible sheaves. $Perv(X)$ is in many aspects better.
- In particular, $Perv(X)$ is preserved by the Verdier duality: $RHom(\mathcal{F}, \omega_X)$ is again a perverse sheaf (up to a shift).

- Note that if $\iota \colon Z \to X$ is a smooth closed subset, then any local system L on Z gives rise to a perverse sheaf $\iota_* L[d]$.
- More generally, if $\iota \colon Z \to X$ is a smooth locally closed subset then any local system L on Z gives rise to a perverse sheaf $IC(L)$ supported on $\overline{Z}$.
- The simple objects of $Perv(X)$ are of the form $IC(L)$, where L is an irreducible local system on a locally closed subset of X.

- Another POV: if X is a complex manifold, we have the category $Conn(X)$ of vector bundles with connections.
- $Conn(X)$ equivalent by Riemann–Hilbert to $Loc(X)$.
- This equivalence can be extended to the equivalence between $D^b(D - mod_{rs}(X))$ and $D^b(X)$, where $D - mod_{rs}(X)$ is the category of holonomic D-modules with regular singularities. Under this equivalence we have

$$D - mod_{rs}(X) \simeq Perv(X).$$

Perverse sheaves on Gr_G

- Recall that $Gr_G = \varinjlim Gr_n$, where Gr_n are projective schemes. We define

$$Perv_{G_O}(Gr_G) := \varinjlim Perv_{G_O}(Gr_n),$$

  where we note that G_O acts on Gr_n via a finite-dimensional quotient.
- We can identify $Perv_{G_O}(Gr_G)$ with the full subcategory of $Perv(G_n)$ consisting of perverse sheaves F such that for every orbit Gr^λ of G_O the perverse sheaf $F|_{Gr^\lambda}$ is a local system (necessarily trivial).

Abelian categories

- $Perv^{G_O}(Gr_G)$ is a semisimple abelian category. Its simple objects correspond to the orbits of G_O on Gr_G. Thus, they are parameterized by X_*.
- $Rep(\check{G})$ is a semisimple abelian category. Its simple objects are parameterized by $X^*(\check{G}) = X_*$.

Tannakian formalism

- A monoidal (a.k.a. tensor) category is a category C with a functor $\otimes \colon C \times C \to C$ that is associative up to a natural isomorphism and an object 1 that is both left and right identity for $\otimes$ up to a natural isomorphism. Some compatibilities are required, e.g., the pentagon diagram.
- A monoidal category is symmetric if there is a commutativity constraint: for any object we are given an isomorphism $c_{X,Y} \colon X \otimes Y \xrightarrow{\simeq} Y \otimes X$ with some compatibilities. (e.g., the hexagon diagram).
- Rigid monoidal category, is a category where each object X has a dual. This is an object Y together with morphisms $1 \to X \otimes Y$ and $Y \otimes X \to 1$, $1 \to Y \times X$, and $X \times Y \to 1$ subject to some compatibilities.

Tannakian formalism

- A neutral Tannakian category over $\mathbb{C}$ is a rigid symmetric abelian tensor $\mathbb{C}$-linear category, such that there exists a $\mathbb{C}$-linear tensor functor (called a fiber functor) to the category of finite dimensional $\mathbb{C}$-vector spaces that is exact and faithful.
- Main example: $Rep(G)$, where G is an affine group scheme.

Theorem

Let C be a neutral Tannakian category $H : C \to Vect_{\mathbb{C}}$ be a fiber functor. Then there is a unique up to isomorphism affine group scheme G and an equivalence of categories $C \simeq Rep(G)$ intertwining H with the forgetful functor.

- Thus, need to equip $Perv^{G_O}(Gr_G)$ with the structure of a neutral Tannakian category (and with a fiber functor).
- The fiber functor is the functor of global cohomology.
- The most intricate structures are the tensor product (called the convolution product) and the commutativity constraint.

Convolution Product

- The idea is to think of objects of $Perv_{G_O}(Gr_G)$ as G_O-biequivariant sheaves on the loop group $G_K = G(\mathbb{C}((t)))$ (recall that $Gr_G = G_K/G_O$). If F_1 and F_2 are two such sheaves, then $F_1 \boxtimes F_2$ is a sheaf on $G_K \times G_K$ and we define

$$F_1 \star F_2 := m_*(F_1 \boxtimes F_2),$$

  where $m\colon G_K \times G_K \to G_K$ is the multiplication.
- Technically, it is more convenient to work with convolution object of ind-finite type $G_K \times_{G_O} Gr_G$.
- One needs to check that the convolution of two perverse sheaves is again a perverse sheaf.

Fusion product and the commutativity constraint

- Recall that $Gr_G = \{(E, s)\}$, where $E \to \mathbb{A}^1$ is a G-torsor, s is a trivialization of E on $\mathbb{A}^1 - 0$.
- A version of Beilinson–Drinfeld Grassmannian:

$$\widetilde{Gr} = \{(x, E, s)\},$$

$x \in \mathbb{A}^1$, $E \xrightarrow{G} \mathbb{A}^1$, s is a trivialization on $\mathbb{A}^1 - 0 - x$.

$$\widetilde{Gr} \to \mathbb{A}^1 \colon (x, E, s) \mapsto x.$$

$$\widetilde{Gr}|_{\mathbb{A}^1 - 0} \approx Gr_G \times (\mathbb{A}^1 - 0) \times Gr_G.$$

$$\widetilde{Gr}|_0 \approx Gr_G.$$

Proposition

- *Start with $F_1, F_2 \in Perv_{G_O}(Gr_G)$.*
- *Consider $F_1 \boxtimes \mathbb{C}_{\mathbb{A}^1 - 0} \boxtimes F_2$ as a perverse sheaf on $\widetilde{Gr}|_{\mathbb{A}^1 - 0}$.*
- *Extend to a perverse sheaf $\tilde{F}$ on $\widetilde{Gr_G}$.*
- *We have $F_1 \star F_2 \simeq \tilde{F}|_0$.*

Maximal torus of $\check{G}$

- For $\nu \in X_*$, let $S^\nu := N_K \cdot t^\nu$, where $N \subset G$ is a maximal unipotent subgroup.
- We have for $F \in Perv_{G_O}(Gr_G)$

$$H^*(Gr_G, F) = \bigoplus_{\nu \in X_*} H^{2\rho(\nu)}(S^\nu, F).$$

- We upgraded H^* to a functor from $Perv_{G_O}(Gr_G)$ to the category of X_*-graded vector spaces.
- The latter category is equivalent to $Rep(\check{T})$.
- This gives a homomorphism

$$\check{T} \to \check{G}.$$

Langlands duality

- Let $D_x = \operatorname{Spec} \mathbb{C}[[t]]$ be the formal disc. Consider the stack Mod classifying triples (E_1, E_2, s), where $E_i \to D_x$ are G-torsors, s is an isomorphism between $E_1|_{\dot{D}_x}$ and $E_2|_{\dot{D}_x}$. Then

$$Perv_{G_O}(Gr_G) = Perv(G_O \setminus G_K / G_O) = Perv(Mod).$$

- Consider the stack Mod' classifying triples (L_1, L_2, s), where $L_i \to D_x$ are $\check{G}$-local systems, s is an isomorphism between $L_1|_{\dot{D}_x}$ and $L_2|_{\dot{D}_x}$. Then

$$Rep(\check{G}) = Coh(B\check{G}) = Coh(Mod').$$

- Local Langlands correspondence:

$$Perv(Mod) \simeq Coh(Mod')$$