

On categories \mathcal{O} for conical symplectic resolutions



Generalities on category \mathcal{O} for conical symplectic resolutions

We fix the base field to be \mathbb{C} . Recall that an affine variety Y is **Poisson** provided it comes equipped with an algebraic Poisson bracket, i.e. a bilinear map

$$\{\cdot, \cdot\} : \Lambda^2 \mathbb{C}[Y] \rightarrow \mathbb{C}[Y],$$

s.t. for any $f, g, h \in \mathbb{C}[Y]$

- $\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$, the Jacobi identity;
- $\{fg, h\} = g\{f, h\} + h\{g, f\}$, the Leibnitz rule.

Let X_0 be a normal Poisson affine variety equipped with an action of the multiplicative group $\mathbb{S} := \mathbb{C}^*$, s.t. the Poisson bracket has a negative degree with respect to this action, i.e.

$$\{\mathbb{C}[X_0]_i, \mathbb{C}[X_0]_j\} \subseteq \mathbb{C}[X_0]_{i+j-d} \text{ with } d \in \mathbb{Z}_{>0}.$$

We assume that $\mathbb{C}[X_0] = \bigoplus_{i \geq 0} \mathbb{C}[X_0]_i$ with $\mathbb{C}[X_0]_0 = \mathbb{C}$ w.r.t. the grading coming from the \mathbb{S} -action (this action will be called the *contracting action*). Geometrically this means that there is a unique fixed point $o \in X_0$ and the entire variety is contracted to this point by the \mathbb{S} -action. Let (X, ω) be a symplectic variety and $\rho : X \rightarrow X_0$ a projective resolution of singularities, which is also a morphism of Poisson varieties. In addition, assume that the action of \mathbb{S} admits a ρ -equivariant lift to X . A pair (X, ρ) as above is called a **conical symplectic resolution**.

Definition. Let (X, ρ) be a conical symplectic resolution. A **quantization** of the affine variety X_0 is an algebra \mathcal{A} together with an isomorphism $\text{gr } \mathcal{A} \xrightarrow{\sim} \mathbb{C}[X_0]$ of graded Poisson algebras. By a quantization of X we understand a sheaf (in the conical topology, i.e. open spaces are Zariski open and \mathbb{S} -stable) of filtered algebras $\tilde{\mathcal{A}}$ (the filtration is complete and separated) together with an isomorphism $\text{gr } \tilde{\mathcal{A}} \xrightarrow{\sim} \mathcal{O}_X$ of sheaves of graded Poisson algebras.

Remark. There are sufficiently many \mathbb{S} -stable open affine subsets. Namely, due to a result of Sumihiro every point x of X_0 has an open affine neighborhood in the conical topology.

Remark. We would like to point out that the algebra $A := \text{gr} \mathcal{A}$ has a natural Poisson bracket. Let $a \in A_i$ and $b \in A_j$ with $\tilde{a} \in \mathcal{A}_{\leq i}$ and $\tilde{b} \in \mathcal{A}_{\leq j}$ any lifts, then the Poisson bracket is given by

$$\{a, b\} := [\tilde{a}, \tilde{b}] + \mathcal{A}_{i+j-2}.$$

Notice that $[\tilde{a}, \tilde{b}] \in \mathcal{A}_{i+j-1}$ since the algebra A is isomorphic to $\mathbb{C}[X_0]$ and hence commutative. It is this bracket that we want to match the original bracket on $\mathbb{C}[X_0]$.

Remark. There is a map from the set of quantizations of X to the second de Rham cohomology $H_{DR}^2(X)$. This map is called the period map and is an isomorphism provided $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$. If this is the case, the quantizations $\tilde{\mathcal{A}}$ are parameterized (up to isomorphism) by the points of $H_{DR}^2(X)$. The quantization corresponding to the cohomology class λ will be denoted by $\tilde{\mathcal{A}}_\lambda$.

Suppose, that X is equipped with a Hamiltonian action of a torus T with finitely many fixed points, i.e. $|X^T| < \infty$. Assume, in addition, that the action of T commutes with the contracting action of \mathbb{S} . A one-parametric subgroup $\nu : \mathbb{C}^* \rightarrow T$ is called *generic* if $X^T = X^{\nu(\mathbb{C}^*)}$. To a generic one-parametric subgroup $\nu : \mathbb{C}^* \rightarrow T$ one can associate a category of modules over the algebra \mathcal{A} defined above, called category $\mathcal{O}_\nu(\mathcal{A})$. Namely, the action of ν lifts to \mathcal{A} and induces a grading on it, i.e. $\mathcal{A} = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}_{i,\nu}$. We denote

$$\mathcal{A}^{\geq 0,\nu} = \bigoplus_{i \geq 0} \mathcal{A}_{i,\nu} \quad (1)$$

$$\mathcal{A}^{\leq 0,\nu} = \bigoplus_{i \leq 0} \mathcal{A}_{i,\nu} \text{ (similarly define } \mathcal{A}^{< 0,\nu}, \mathcal{A}^{> 0,\nu} \text{) and} \quad (2)$$

$$\mathcal{C}_\nu(\mathcal{A}) := \mathcal{A}^{\geq 0,\nu} / (\mathcal{A}^{\geq 0,\nu} \cap \mathcal{A}\mathcal{A}^{> 0,\nu}) = \mathcal{A}_0 / \bigoplus_{i > 0} \mathcal{A}_{-i}\mathcal{A}_i. \quad (3)$$

Let $\mathcal{A}\text{-mod}$ be the category of finitely generated \mathcal{A} -modules.

Definition. The category $\mathcal{O}_\nu(\mathcal{A})$ is the full subcategory of $\mathcal{A}\text{-mod}$, on which $\mathcal{A}^{\geq 0, \nu}$ acts locally finitely.

Recall that if \mathcal{R} is a commutative Noetherian ring and $X = \text{Spec}\mathcal{R}$, then one has an equivalence of abelian categories:

$$\mathcal{R}\text{-mod} \underset{\Gamma}{\overset{\text{Loc}}{\rightleftarrows}} \text{Coh}(X), \quad (1)$$

where Γ and Loc are the functor of global sections and localization respectively.

Definition. An \mathcal{A}_λ -module M is called **coherent** provided there is a global complete and separated filtration on M , s.t. grM is a coherent \mathcal{O}_X -module. The category of coherent \mathcal{A}_λ -modules will be denoted by $\text{Coh}(\mathcal{A}_\lambda)$.

The noncommutative analogue of the equivalence on the previous slide is

$$\mathcal{A}_\lambda\text{-mod} \underset{\Gamma_\lambda}{\overset{\text{Loc}_\lambda}{\rightleftarrows}} \text{Coh}(\widetilde{\mathcal{A}}_\lambda)$$

and has a weaker (derived) form:

$$D^b(\mathcal{A}_\lambda\text{-mod}) \underset{R\Gamma_\lambda}{\overset{L\text{Loc}_\lambda}{\rightleftarrows}} D^b(\text{Coh}(\widetilde{\mathcal{A}}_\lambda)).$$

Definition. *If the functors Γ_λ and Loc_λ are mutually inverse equivalences, we say that **abelian localization holds** for λ and if $R\Gamma_\lambda$ and $L\text{Loc}_\lambda$ are quasi-inverse equivalences (between the bounded derived categories) that **derived localization holds**.*

Definition. We have the standardization and costandardization functors Δ_ν and $\nabla_\nu : \mathcal{C}_\nu(\mathcal{A}_\lambda)\text{-mod} \rightarrow \mathcal{O}_\nu(\mathcal{A}_\lambda)$ given by

$$\Delta_\nu(N) := \mathcal{A}_\lambda \otimes_{\mathcal{A}_\lambda^{\geq 0}} N$$

$$\nabla_\nu(N) := \text{Hom}_{\mathcal{C}_\nu(\mathcal{A}_\lambda)}(\mathcal{A}_\lambda(n, \ell) / \mathcal{A}_\lambda^{< 0} \mathcal{A}_\lambda, N).$$

Remark. For generic λ and ν :

1. $\mathcal{C}_\nu(\mathcal{A}_\lambda) = \mathbb{C}[X^T]$;
2. if there are finitely many fixed points for the action of ν , then the category \mathcal{O}_ν is highest weight with standard and costandard objects indexed by $p_i \in X^T$ and the partial order being the attraction order for ν .

Example. Let \mathfrak{g} be a simple Lie algebra with Borel subalgebra \mathfrak{b} and Cartan subalgebra \mathfrak{h} . In order to fit the classical BGG category \mathcal{O} in this framework, one needs to consider the Springer resolution $X = T^*(G/B) \rightarrow \mathcal{N} = X_0$ of the nilpotent cone $\mathcal{N} \subset \mathfrak{g}^*$. Recall that an element $x \in \mathfrak{g}$ is called nilpotent if the operator $\text{ad}_x^* : \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is nilpotent and \mathcal{N} is the set of all nilpotent elements of \mathfrak{g}^* . The nilcone \mathcal{N} is a Poisson variety w.r.t. the Kirillov-Kostant-Souriau bracket and the symplectic leaves in \mathcal{N} are the coadjoint orbits. The tori are the maximal torus $T \subset GL(V)$ and $\mathbb{S} := \mathbb{C}^*$ acting by inverse scaling on the cotangent fibers. Let $\mu : Z(\mathfrak{g}) \rightarrow \mathbb{C}$ be a central character, then the block $\mathcal{O}_\mu \subset \mathcal{O}$ consists of finitely generated $U(\mathfrak{g})$ -modules for which $U(\mathfrak{b})$ acts locally finitely, $U(\mathfrak{h})$ semisimply and the center with generalized character μ . Pick a generic one-parameter subgroup $\nu(\mathbb{C}^*) \subset T$, s.t. \mathfrak{b} is spanned by elements with positive $\nu(\mathbb{C}^*)$ -weights. Let $U(\mathfrak{g})_\mu = U(\mathfrak{g})/\mathcal{I}_\mu$ with \mathcal{I}_μ the ideal generated by $z - \mu(z)$ for $z \in Z(\mathfrak{g})$ be the central reduction of $U(\mathfrak{g})$ w.r.t the central character μ .

We want to show that $U(\mathfrak{g})_\mu$ is a quantization of the nilcone \mathcal{N} . One can explicitly describe the Poisson bracket on $\mathbb{C}[\mathcal{N}]$ descending from $U(\mathfrak{g})_\mu$. Recall that according to the PBW theorem $grU(\mathfrak{g})$ is isomorphic to $S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$. Moreover, the Harish Chandra theorem asserts that $Z(\mathfrak{g})$ is isomorphic to $S(\mathfrak{h})^W = \mathbb{C}[\mathfrak{h}^*]^W$. Here W is the Weyl group acting on \mathfrak{h}^* via $w \cdot \mu = w(\mu + \rho) - \rho$, where ρ is half the sum of all positive roots. Combining these results allows to show the isomorphism of algebras $grU(\mathfrak{g})_\mu \simeq \mathbb{C}[\mathcal{N}]$. Let x_1, \dots, x_n be a basis of \mathfrak{g} and $c_{ij}^k \in \mathbb{C}$ the structure constants given by $[x_i, x_j] = \sum_{k=1}^n c_{ij}^k x_k$. The Poisson bracket on $\mathcal{N} \subset \mathfrak{g}^*$ becomes the restriction of the bracket on \mathfrak{g}^* given by

$$\{f, g\} = \sum_{k=1}^n c_{ij}^k x_k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}, \text{ for } f, g \in \mathbb{C}[\mathfrak{g}^*].$$

The last equality can be more conveniently rewritten as

$$\{f, g\}(\xi) = \langle \xi, [d_\xi f, d_\xi g] \rangle,$$

where $\xi \in \mathfrak{g}^*$, $d_\xi f \in \mathfrak{g}^{**} \simeq \mathfrak{g}$ stands for the differential of f at ξ and $[\cdot, \cdot]$ denotes the Lie bracket on \mathfrak{g} . This is exactly the Kirillov-Kostant-Souriau bracket on the nilcone \mathcal{N} .

Next we want to compare the categories $\mathcal{O}_\nu(U(\mathfrak{g})_\mu)$ and \mathcal{O}_μ . The difference in the requirements for an object $M \in U(\mathfrak{g})_\mu\text{-mod}$ to be in $\mathcal{O}_\nu(U(\mathfrak{g})_\mu)$ or \mathcal{O}_μ is that for the former containment $Z(\mathfrak{g})$ must act on M with the **fixed character** μ , while for the latter the center acts by the **generalized character** μ and the **action of** $U(\mathfrak{h})$ on M has to be **semisimple**. In case μ is regular ($\langle \mu + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{\leq 0}$ for all positive roots α) these conditions are interchangeable, i.e. one gets an equivalent category by dropping one condition and adding the other (nontrivial result of Soergel) and, hence, the categories $\mathcal{O}_\nu(U(\mathfrak{g})_\mu)$ and \mathcal{O}_μ are equivalent.

Finally, let $\mathcal{D}_\mu(G/B)$ stand for the category of μ -twisted \mathcal{D} -modules on the flag variety G/B . Then one has an equivalence

$$U(\mathfrak{g})_\mu\text{-mod} \underset{\Gamma}{\overset{\text{Loc}}{\rightleftarrows}} \mathcal{D}_\mu(G/B)\text{-mod}$$

for regular μ , this is the Beilinson-Bernstein theorem, while

$$D^b(U(\mathfrak{g})_\mu\text{-mod}) \underset{\text{R}\Gamma}{\overset{\text{LLoc}}{\rightleftarrows}} D^b(\mathcal{D}_\mu(G/B)\text{-mod})$$

is an equivalence provided $\langle \mu + \rho, \alpha^\vee \rangle \neq 0$.

A few examples of conical symplectic resolutions and properties of their quantizations and categories \mathcal{O} .

Example 1. *Let $\Gamma \subset SL_2(\mathbb{C})$ be a nontrivial finite subgroup. Take $Y_0 = \mathbb{C}^2/\Gamma$ and the crepant resolution $\rho : Y \rightarrow Y_0$. The action of \mathbb{S} is induced by the inverse of the diagonal action on \mathbb{C}^2 , and has weight $d = 2$. In case $\Gamma = \mathbb{Z}/k\mathbb{Z}$, we can find a Hamiltonian $T \simeq \mathbb{C}^*$ -action, where T is the group of symplectomorphisms of Y that commute with the action of \mathbb{S} .*

Example 2. *For a reductive algebraic group G and a Borel subgroup B , take X to be the cotangent bundle of flag variety $T^*(G/B)$ and X_0 to be the affinization of X . The map ρ is the Springer resolution. The action of \mathbb{S} is via inverse scaling on the cotangent fibers, and $d = 1$, while $T \subset G$ is the maximal torus.*

Example 3. *Hypertoric varieties associated to simple, unimodular hyperplane arrangements. These varieties admit an \mathbb{S} -action with $d = 1$ if and only if the arrangement has a bounded chamber; they always admit an action with weight $d = 2$. This was the first class of conical symplectic resolutions for which the unified definition of categories \mathcal{O} in the context outlined above was given.*

Main Example. *Nakajima quiver varieties. These varieties admit an action of \mathbb{S} with $d = 1$ if and only if the quiver has no loops; they always admit an action with weight $d = 2$.*

Remark. *The last class of examples overlaps with the preceding ones. Namely, the first example is a special case of quiver varieties, where the underlying graph of the quiver is the extended Dynkin diagram corresponding to Γ . The varieties that appear in the second example can be realized as quiver varieties if the group is of type A . Finally, a hypertoric variety is a quiver variety if and only if a certain technical condition on the hyperplane arrangement is satisfied.*

Nakajima quiver varieties

Let $Q = (Q_0, Q_1)$ be a finite quiver, i.e. a directed graph with finitely many vertices enumerated by the set Q_0 and finitely many edges enumerated by Q_1 . Each edge is uniquely determined by the pair of vertices it connects, which we will denote by $t(a)$ and $h(a)$ standing for 'tail' and 'head'. Consider two dimension vectors $\mathbf{v} = (v_1, \dots, v_n)$ and $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{Z}_{\geq 0}^n$, where n is the cardinality of Q_0 and form a vector space

$$R = \bigoplus_{a \in Q_1} \text{Hom}_{\mathbb{C}}(V_{ta}, V_{ha}) \oplus \bigoplus_{s \in Q_0} \text{Hom}_{\mathbb{C}}(V_s, W_s).$$

Remark. *The dimension vector \mathbf{w} is often referred to as framing.*

Notice that the space T^*R is symplectic and naturally identified with

$$\bigoplus_{a \in Q_1} (\mathrm{Hom}_{\mathbb{C}}(V_{ta}, V_{ha}) \oplus \mathrm{Hom}_{\mathbb{C}}(V_{ha}, V_{ta})) \oplus \bigoplus_{s \in Q_0} (\mathrm{Hom}_{\mathbb{C}}(V_s, W_s) \oplus \mathrm{Hom}_{\mathbb{C}}(W_s, V_s)).$$

We will use the notation $(\mathbf{x}, \bar{\mathbf{x}}, \mathbf{i}, \mathbf{j})$ to represent a point $p \in T^*R$, where

$$\begin{aligned} \mathbf{x} &= (x_a \in \mathrm{Hom}_{\mathbb{C}}(V_{ta}, V_{ha}))_{a \in Q_1}, \\ \bar{\mathbf{x}} &= (x_a^* \in \mathrm{Hom}_{\mathbb{C}}(V_{ha}, V_{ta}))_{a \in Q_1}, \\ \mathbf{i} &= (i_s \in \mathrm{Hom}_{\mathbb{C}}(V_s, W_s))_{s \in Q_0} \text{ and} \\ \mathbf{j} &= (j_s \in \mathrm{Hom}_{\mathbb{C}}(W_s, V_s))_{s \in Q_0}. \end{aligned}$$

The reductive group $G := \prod_{i=1}^n GL(V_i)$ naturally acts on R . We are interested in the induced Hamiltonian action of G on T^*R . The corresponding moment map $\mu : T^*R \rightarrow \mathfrak{g}^*$ is given by

$$\mu(\mathbf{x}, \bar{\mathbf{x}}, \mathbf{i}, \mathbf{j}) = \sum_{a \in Q_1} (x_a x_{a^*} - x_{a^*} x_a) - \sum_{s \in Q_0} j_s i_s. \quad (1)$$

To define the Nakajima quiver variety $\mathcal{M}^\theta(Q, \mathbf{v}, \mathbf{w})$, we need to choose some character θ of G . Such θ is uniquely determined by an n -tuple of integers, i.e. by $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{Z}^n$ we understand the character θ as a map $(g_1, \dots, g_n) \mapsto \prod_{s=1}^n \det(g_s)^{\theta_s}$, where $g_s \in GL(V_s)$.

Definition. The GIT quotient $\mathcal{M}_0^\theta(Q, \mathbf{v}, \mathbf{w}) := \mu^{-1}(0)^{\theta-ss} //^\theta G$ is called the Nakajima quiver variety with parameter θ .

Remark. The affine variety $\mathcal{M}_0^0(Q, \mathbf{v}, \mathbf{w})$ is $\mu^{-1}(0)/G = \text{Spec}(\mathbb{C}[T^*R]/I)^G$, where $I := \{\lambda(\xi), \xi \in \mathfrak{g}\}$ is the ideal generated by the image of \mathfrak{g} under the comoment map.

Theorem. There is a projective morphism $\rho : \mathcal{M}_0^\theta(Q, \mathbf{v}, \mathbf{w}) \rightarrow \mathcal{M}_0^0(Q, \mathbf{v}, \mathbf{w})$, which is a (conical) symplectic resolution of singularities for generic θ .

Remark. *An application of the Hilbert-Mumford criterion shows that :*

- 1. if $\theta_t > 0 \forall t$, then a quadruple $(\mathbf{x}, \bar{\mathbf{x}}, \mathbf{i}, \mathbf{j}) \in \mu^{-1}(0)$ is θ -semistable if and only if for any collection of vector subspaces $S = (S_t)_{t \in Q_0} \subset V = (V_t)_{t \in Q_0}$, which is stable under the maps $\mathbf{x}, \bar{\mathbf{x}}$, we have*

$$S_t \subset \ker(i_t) \quad \forall t \in Q_0 \Rightarrow S = 0,$$

- 2. if $\theta_t < 0 \forall t$, then a quadruple $(\mathbf{x}, \bar{\mathbf{x}}, \mathbf{i}, \mathbf{j}) \in \mu^{-1}(0)$ is θ -semistable if and only if for any collection of vector subspaces $S = (S_t)_{t \in Q_0} \subseteq V = (V_t)_{t \in Q_0}$, which is stable under the maps $\mathbf{x}, \bar{\mathbf{x}}$, we have*

$$S_t \supset \operatorname{im}(j_t) \quad \forall t \in Q_0 \Rightarrow S = V.$$

Next we describe quantizations of $\mathcal{M}_0^0(Q, \mathbf{v}, \mathbf{w})$. Denote the ring of differential operators on R by $D(R)$.

Definition. A G -equivariant linear map $\Phi : \mathfrak{g} \rightarrow D(R)$, satisfying $[\Phi(x), a] = x_R(a)$ for any $x \in \mathfrak{g}$ and $a \in D(R)$ is called a **quantum comoment map**.

Remark. The quantum comoment map Φ is defined up to adding a character $\lambda : \mathfrak{g} \rightarrow \mathbb{C}$.

Next we take a character λ of \mathfrak{g} and consider the quantization

$$\mathcal{A}_\lambda(n, \ell) := (D(R)/D(R)\mathcal{I}_\lambda)^G,$$

where $\mathcal{I}_\lambda = \langle \Phi(x) - \lambda(x) \rangle_{x \in \mathfrak{g}}$.

The filtration on $\mathcal{A}_\lambda(n, \ell)$ is induced from the Bernstein filtration on $D(R)$ (here $\deg R = \deg R^* = 1$). Recall that $\mathbb{C}[\mathcal{M}_0^0(Q, \mathbf{v}, \mathbf{w})] = (\mathbb{C}[T^*R]/I)^G$, where $I := \{\lambda(\xi), \xi \in \mathfrak{g}\}$ is the ideal generated by the image of \mathfrak{g} under the comoment map. The surjectivity of the natural map $\mathbb{C}[\mathcal{M}_0^0(Q, \mathbf{v}, \mathbf{w})] \rightarrow \text{gr } \mathcal{A}_\lambda(n, \ell)$ follows from the containment $I \subset \text{gr } \mathcal{I}_\lambda$.

Definition. Let V be a finite dimensional vector space over a field k . Let $\mathcal{R} \subset V \otimes V$ be a subspace and denote by $\mathcal{I}_{\mathcal{R}}$ the ideal in the tensor algebra $TV := k \oplus V \oplus V \otimes V \oplus \dots$ generated by \mathcal{R} . The algebra $A := TV/\mathcal{I}_{\mathcal{R}}$ is called a *quadratic algebra*. Its *quadratic dual* is the algebra $A^! \simeq TV^*/\mathcal{I}_{\mathcal{R}^\perp}$, where

$$\mathcal{I}_{\mathcal{R}^\perp} \subset V^* \otimes V^*$$

is the dual ideal.

Examples. 1. If $\mathcal{R} = \{0\}$ then $A = TV$ and $A^! = TV^*/(V^* \otimes V^*)$.

2. If $A = S^\bullet V$ is the symmetric algebra over V , then $A^! = \Lambda^\bullet V^*$ is the exterior algebra over the dual vector space.

Recall that for $G_1 \subset G[t^{-1}]$, the kernel of the evaluation map $G[t^{-1}] \rightarrow G$ and $\mu \prec \lambda$ two dominant coroots, the intersection $Gr_\mu^\lambda := G_1 \cdot t^\lambda \cap \overline{Gr}_\lambda$ is called the **Luzstig slice**. It is an affine Poisson variety. The truncated shifted Yangian Y_μ^λ is an algebra which quantizes $\overline{Gr}_\mu^\lambda$. The algebra Y_μ^λ contains a polynomial subalgebra (the analogue of $\mathcal{C}_\nu(\mathcal{A})$ from slide 6) and thus the notions of weight modules and category \mathcal{O} make sense.

Theorem. *For a generic choice of integral parameters (\mathbf{v}, \mathbf{w}) , the quadratic dual of category \mathcal{O} for Y_μ^λ is the category \mathcal{O} for the quiver variety $\mathcal{M}_0^0(Q, \mathbf{v}, \mathbf{w})$. If $\mathbf{v} = \lambda$ is a sum of minuscule weights, then both categories \mathcal{O} are Koszul, and they are Koszul dual to each other.*