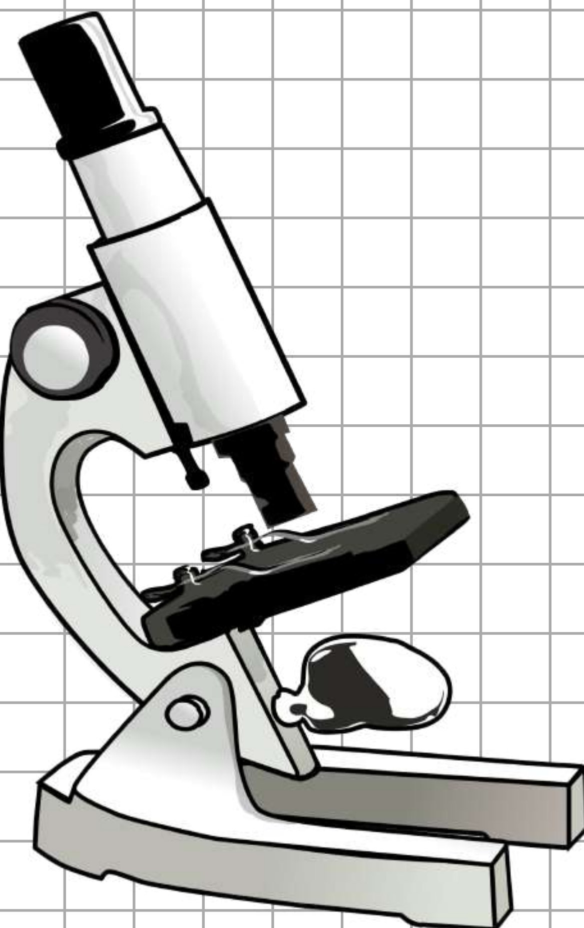
Intro to affine Grassmannians.



$\mathcal{O} = \mathbb{C}[[t]]$ - power series

$K = \mathrm{Frac}(\mathcal{O}) = \mathbb{C}((t))$ - formal Laurent polynomials.

Def. An $\mathcal{O}$-lattice in K^n is a projective finitely generated $\mathcal{O}$-submodule Λ, s.t.

$$\Lambda \otimes_{\mathcal{O}} K \simeq K^n.$$

Such lattices are points in the affine Grassmannian.

Goal: endow with topology!

Let $Gr_N := \{\Lambda \mid t^{-N}\Lambda_0 \supset \Lambda \supset t^N \Lambda_0\}$, where $\Lambda_0 = \mathcal{O}^n$.

<u>Rmk:</u> $Gr_N \subseteq Gr_{N+1} \subset \ldots$

$$Gr = \varinjlim Gr_N$$

Notice that $t^{-N}\Lambda_0 / t^N \Lambda_0 \simeq \mathbb{C}^{2nN}$

There is a map $\varphi : Gr_N \hookrightarrow Gr(2nN)$

$$Gr(2nN) = \bigsqcup_{k \in \{1,2,\ldots,2nN-1\}} Gr(k, 2nN)$$

$$\varphi(\Lambda) = \Lambda / t^N \Lambda_0$$

<u>Rmk.</u> φ is not surjective, since to be an $\mathcal{O}$-submodule, a subspace must be t-stable.

Recall: $Gr(2nN)$ is a projective var-ty (via Plücker embedding), t-stability is a closed condition, so we get an induced structure of proj. var-ty on Gr_N.

<u>Example.</u> $N=0$, $Gr_0 = \{\Lambda \mid \Lambda_0 \supseteq \Lambda \supseteq \Lambda_0\} =$

$= \Lambda_0 = pt.$

Conclusion: $\mathcal{U}: Gr_N \hookrightarrow Gr(2nN)$ is a closed embedding, giving Gr_N a structure of proj. scheme and $Gr = \varinjlim Gr_N$ the structure of ind-proj. scheme.

Cartan decomposition / Affine Schubert cells.

Recall: G - classical Lie group $\supset B$ - Borel subgroup

Bruhat decomposition:

$$G = \bigsqcup_{w \in W} BwB$$

Example. $G = GL_n$, $B = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$, $W = S_n$.

For instance, $(12) = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & 0 \\ & & 1 & \\ 0 & & & \ddots \\ & & & & 1 \end{pmatrix}$.

Acting by row and column transformations, we can bring any matrix to a unique permutation matrix.

Cartan decomposition:

$$G(K) = \coprod_{\substack{\lambda\text{-dominant}\\ \text{coweights}}} G(\mathcal{O})\, t^{\lambda}\, G(\mathcal{O}).$$

In case $G = GL_n$, $t^{\lambda} = \begin{pmatrix} t^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & t^{\lambda_n} \end{pmatrix}$ with $\lambda_i \in \mathbb{Z}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

<u>Proof</u>: Gauss-Jordan elimination (Smith normal form)

<u>Rmk</u>. $G(\mathcal{O})$ is the analog of $P \subset G$ (maximal parabolic) and is called <u>parahoric</u> = 'Iwahori + parabolic'

Analog of B is I:

$$\pi: G(\mathcal{O}) \longrightarrow G, \quad I = \pi^{-1}(B).$$
$$t \mapsto 0$$

I stabilizes a full flag of lattices

'Old' Grassmannian $P = \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) \subset G$ $Gr \cong G/P$.

$B \curvearrowright G/P \rightsquigarrow$ Schubert cells

$\|$

(closures of) B-orbits.

'New' Grassmannian:

$$G(\mathcal{O}) \curvearrowright G(K)/G(\mathcal{O})$$

Affine Schubert cells are

$X_\lambda = G(\mathcal{O}) \cdot t^\lambda$.

<u>Fact.</u> $\overline{X_\lambda} = \bigsqcup_{\mu \preceq \lambda} X_\mu$, μ is dominant.

$\mu \prec \lambda$ means that $\mu - \lambda \in X_+$ (positive wt).

For GL_n, $\mu \preceq \lambda$ means $\mu_1 \leq \lambda_1$

$\mu_1 + \mu_2 \leq \lambda_1 + \lambda_2$

$\vdots$

$\mu_1 + \dots + \mu_n \leq \lambda_1 + \dots + \lambda_n$

<u>Rmk.</u> $\overline{X_\lambda}$ is closed if λ is a <u>minuscule wt</u> (not greater than any $\mu \in X_+$).

Example. The minuscule wts for GL_n are

$$\lambda_k = (\underbrace{1, 1, \ldots, 1}_{k}, \underbrace{0, 0, \ldots, 0}_{n-k})$$

Prop-n. $\overline{X}_{\lambda_k} = X_{\lambda_k} \cong Gr(n-k, n)$.

Indeed, this follows from a computation

$$\begin{pmatrix} P_{11}(t) & \cdots & P_{1n}(t) \\ \vdots & & \vdots \\ P_{n1}(t) & \cdots & P_{nn}(t) \end{pmatrix} \begin{pmatrix} t & & & & & 0 \\ & \ddots & & & & \\ & & t & & & \\ & & & 1 & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{pmatrix}$$

$$\shortparallel$$

$$\begin{pmatrix} tP_{11}(t) & \cdots & tP_{1k}(t) & \boxed{P_{1k+1}(t) & \cdots & P_{1n}(t)} \\ \vdots & & & \vdots & & \vdots \\ tP_{n1}(t) & \cdots & tP_{nk}(t) & P_{nk+1}(t) & \cdots & P_{nn}(t) \end{pmatrix}$$

$$\underset{\substack{\uparrow \\ GL_n(\mathcal{O})}}{t^{\lambda_k} \cdot Q(t)} = \begin{pmatrix} tQ_{11}(t) & \cdots & tQ_{1n}(t) \\ \vdots & & \vdots \\ tQ_{k1}(t) & \cdots & tQ_{kn}(t) \\ Q_{k+1 1}(t) & \cdots & Q_{k+1 n}(t) \\ \vdots & & \vdots \\ Q_{n1}(t) & \cdots & Q_{nn}(t) \end{pmatrix}$$

Conclusion: the action of $G(\mathcal{O})$ factors through the action of $t \cdot G(\mathcal{O})$

$$g \in G(\mathcal{O})$$
$$\| $$
$$G \ni \underline{g_0} + t\, g_1(t).$$

$$G(\mathcal{O}) = G + \underbrace{t \cdot G(\mathcal{O})}_{\text{acts trivially}}$$

$$\cup$$

$$P = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$$

We get that $X_{\lambda_k} \simeq G/P \simeq Gr(n-k, n)$.

The stabilizer of t^λ for left $G(\mathcal{O})$-action is $G(\mathcal{O}) \cap t^\lambda \cdot G(\mathcal{O}) \cdot t^{-\lambda}$

In the example above $t^{\lambda_k} G(\mathcal{O}) t^{-\lambda_k} =$

$$= \begin{pmatrix} t & & & & \\ & \ddots & & 0 & \\ & & t & & \\ & 0 & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & & \vdots \\ p_{n1}(t) & \cdots & p_{nn}(t) \end{pmatrix} \begin{pmatrix} t^{-1} & & & & \\ & \ddots & & 0 & \\ & & t^{-1} & & \\ & 0 & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} =$$

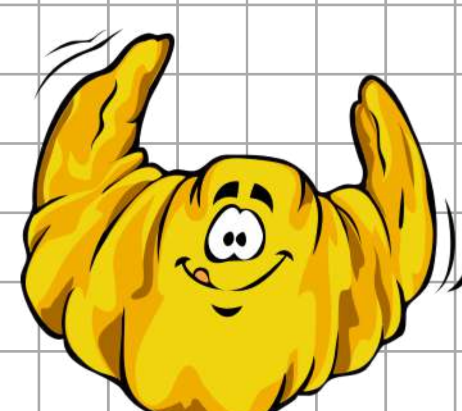
$$= \left(\begin{array}{ccc|ccc} p_{11}(t) & \cdots & p_{1k}(t) & tp_{1k+1}(t) & \cdots & tp_{1n}(t) \\ \vdots & & \vdots & \vdots & & \vdots \\ p_{k1}(t) & \cdots & p_{kk}(t) & tp_{kk+1}(t) & \cdots & tp_{kn}(t) \\ \hline t^{-1}p_{k+11}(t) & \cdots & t^{-1}p_{k+1k}(t) & p_{k+1k+1}(t) & \cdots & p_{k+1n}(t) \\ \vdots & & \vdots & \vdots & & \vdots \\ t^{-1}p_{n1}(t) & \cdots & t^{-1}p_{nk}(t) & p_{nk+1}(t) & \cdots & p_{nn}(t) \end{array}\right);$$

After intersecting with $G(\vartheta)$, we get that $\mathrm{Stab}_{t^{\lambda_k}}$ consists of matrices of the form

$$g = \left(\begin{array}{ccc|ccc} p_{11}(t) & \cdots & p_{1k}(t) & tp_{1k+1}(t) & \cdots & tp_{1n}(t) \\ \vdots & & \vdots & \vdots & & \vdots \\ p_{k1}(t) & \cdots & p_{kk}(t) & tp_{kk+1}(t) & \cdots & tp_{kn}(t) \\ \hline \vdots & & \vdots & \vdots & & \vdots \\ p_{n1}(t) & \cdots & p_{nk}(t) & p_{nk+1}(t) & \cdots & p_{nn}(t) \end{array}\right) \in G(\vartheta)$$

$$G(\vartheta)/\mathrm{Stab}_{t^{\lambda_k}} \cong G/P, \quad P = \left(\begin{array}{c|c} * & 0 \\ \hline * & * \end{array}\right)$$

Coffee break



Recall: $\rho = \frac{1}{2}\sum_{\alpha\in\Phi_+}\alpha$

Prop-n. $\dim X_\lambda = (2\rho, \lambda)$.

Pf: $X_\lambda \simeq G(\mathcal{O})/(G(\mathcal{O})\cap t^\lambda G(\mathcal{O})t^{-\lambda})$, hence, X_λ is smooth and the dimension of X_λ equals the dimension of X_λ at any point x:

$$\dim T_x(X_\lambda) = \dim\left(\mathfrak{g}(\mathcal{O})/(\mathfrak{g}(\mathcal{O})\cap Ad_{t^\lambda}(\mathfrak{g}(\mathcal{O})))\right) =$$

$$\dim\left(\bigoplus_{\alpha\in\Phi_+}\mathfrak{g}_\alpha(\mathcal{O})/t^{(\lambda,\alpha)}\mathfrak{g}_\alpha(\mathcal{O})\right) = \sum_{\alpha\in\Phi_+}(\alpha,\lambda) = (2\rho,\lambda).$$

Rmk. λ is dominant, so $(\alpha,\lambda)<0$ for any $\alpha\in\Phi_-$ and $\mathfrak{g}_\alpha(\mathcal{O})/t^{(\lambda,\alpha)}\cdot\mathfrak{g}_\alpha(\mathcal{O}) = 0$.

Example. $\lambda_k = (\underbrace{1,1,\dots,1}_{k},\underbrace{0,\dots,0}_{n-k})$, a minuscule coweight for $G = GL_n$.

As $2\rho = \sum_{1\le i<j\le n}\varepsilon_i-\varepsilon_j = \sum_{1\le i<j\le k}\varepsilon_i-\varepsilon_j + \sum_{1\le i\le k<j\le n}\varepsilon_i-\varepsilon_j +$
$+\sum_{k<i<j\le n}\varepsilon_i-\varepsilon_j$, where $\varepsilon_i(e_j) = \delta_{i,j}$,

we have $(2\rho, \lambda_k) = \Big(\sum_{1\le i\le k<j\le n} \varepsilon_i - \varepsilon_j, \lambda_k\Big) = \underset{\dim Gr(n-k,n).}{\underset{\parallel}{k(n-k)}}$

Nilcone inside affine Grassmannian.

Def. The subvariety $\mathcal{N} = \{A \in \mathfrak{g} \mid A^n = 0\}$ is called the nilpotent cone.

Rmk. The definition above works for $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{g} \hookrightarrow \mathfrak{gl}_n$.

Example. $\mathfrak{g} = \mathfrak{sl}_2 = \left\{ \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \right\}$

$A \in \mathfrak{g}$ is nilpotent $\Leftrightarrow A^2 = 0 \xLeftrightarrow[\text{thm}]{\text{Cayley-Hamilton}}$ $\chi_A(t) = t^2$. As $\mathfrak{sl}_2$ consists of traceless matrices, $\chi_A(t) = t^2 \Leftrightarrow \det A = -x^2 - yz = 0$, i.e.

$$\mathcal{N} \simeq \mathbb{C}[x,y,z]/(x^2+yz) \text{ is a cone}$$

This is where the name 'nilpotent cone' or 'nilcone' comes from.

If $\mathfrak{g} = \mathfrak{gl}_n$, then an operator $A \in \mathcal{N}$ iff $\chi_A(t) = t^n + a_{n-1}t^{n-1} + \ldots + a_1 t + a_0 = t^n$, i.e. the coefficients $a_0, a_1, \ldots, a_{n-1}$ (which are polynomials in the matrix entries of A all vanish).

This allows to conclude

$$\dim \mathcal{N} = \dim \mathfrak{g} - n = n^2 - n.$$

The following construction is attributed to G. Lusztig.

Let $\mu = (n, \underbrace{0, 0, \ldots, 0}_{})$. It is not hard to check that $\overline{X_\mu} = \overline{G(\mathcal{O}) \cdot t^\mu} \supset \{\Lambda \mid \Lambda_0 \supset \Lambda \supset t^n \Lambda_0, \dim(\Lambda_0/\Lambda) = n\}$.

Consider the map

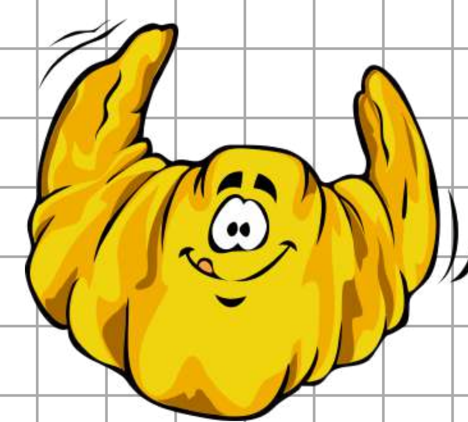
$$\begin{aligned} \Psi : \mathcal{N} &\hookrightarrow \overline{X_\mu} \\ A &\mapsto \Lambda_0 / (t-A)\Lambda_0 . \end{aligned}$$

<u>Rmk.</u> This is the same construction as the one used in the proof of existence of Jordan canonical form: given a matrix $A \in \mathfrak{gl}_n$ and $V = \mathbb{C}^n$, we make V into a $\mathbb{C}[X]$-module

with the action of $f(x) \in \mathbb{C}[x]$ being via $f(A)$.

Rmk. Notice that $\dim \overline{X}_\mu = (2\rho, \mu) = n(n-1) = \dim \mathcal{V}$, hence, Ψ is an open embedding.

Coffee break



Valuation.

Let $\Lambda = \mathrm{span}_{\mathcal{O}}\{v_1, \dots, v_n\}$ be a lattice, then
$\det\left[v_1 \middle| v_2 \middle| \dots \middle| v_n\right] \in K^\times$.

Define the map $\mathrm{val}: Gr \to \mathbb{Z}$ via

$$\mathrm{val}(\Lambda) = \min\{n \mid t^n \text{ occurs in } \det(\text{basis})\}.$$

Properties:
1. Independent of the choice of basis
2. Constant on left $G(\mathcal{O})$-orbits. (Schubert cells)

Reason: any matrix $g \in G(\mathcal{O})$ has $\det g \in \mathbb{C}[[t]]^{\times}$ (is invertible), i.e. $\det g = a_0 + a_1 t + \dots$ with $a_0 \neq 0$. It follows that multiplication by g (left or right) does not change the minimal power of t in the determinant.

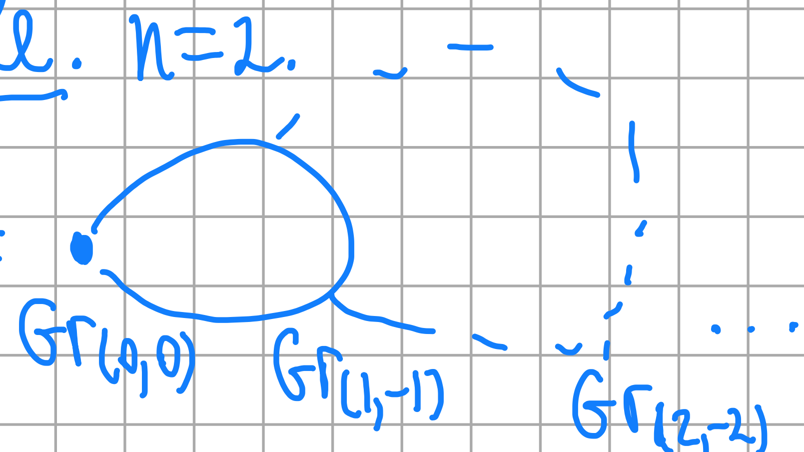
Complete picture for GL_n

As shown above, we have a map

$$\mathrm{val}: \left\{ \begin{array}{c} \text{connected com-} \\ \text{ponents of } Gr \end{array} \right\} \longrightarrow \mathbb{Z}$$

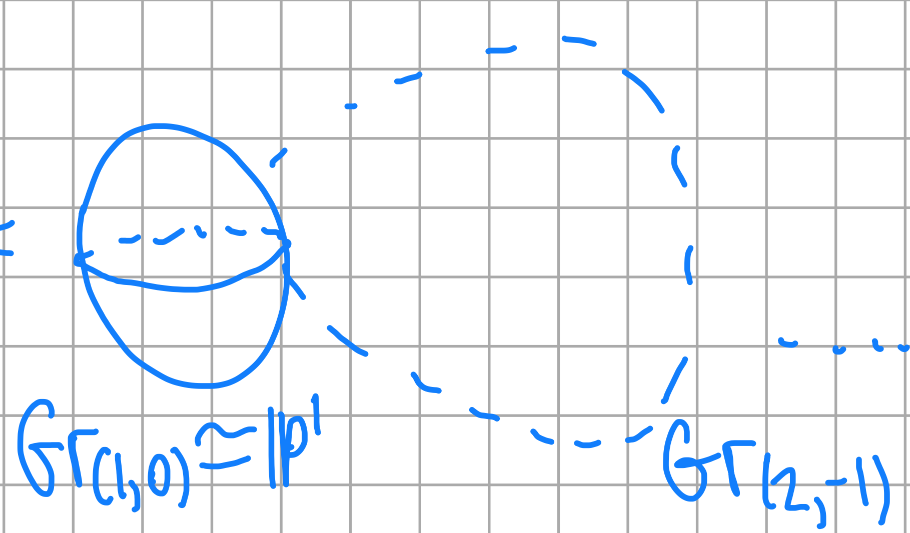
Example. $n=2$.

$\mathrm{val}^{-1}(0) =$



$\dim Gr_{(m,-m)} =$
$= ((1,-1), (m,-m)) = 2m$

$\mathrm{val}^{-1}(1) =$



$\dim Gr_{(m,-m+1)} = 2m+1.$

($(1,0)$ is a minuscule weight)

<u>Rmk.</u> $val^{-1}(2k)$ is $GL_2(\mathcal{O})$-equivariantly isomorphic to $val^{-1}(0)$ and $val^{-1}(2k+1)$ is $GL_2(\mathcal{O})$-equivariantly isomorphic to $val^{-1}(0)$ for any $k \in \mathbb{Z}$.
The isomorphisms are given by multiplication by the matrix $\begin{pmatrix} t^k & 0 \\ 0 & t^k \end{pmatrix}$ and its inverse $\begin{pmatrix} t^{-k} & 0 \\ 0 & t^{-k} \end{pmatrix}$.
In other words, we get the bijection

$$\left\{ \begin{array}{l} \text{iso-classes of connec-} \\ \text{ted components of } Gr \\ (\text{for } GL_2) \end{array} \right\} \longleftrightarrow \mathbb{Z}_2$$

parity of valuation

Similarly one gets

$$\left\{ \begin{array}{l} \text{iso-classes of connec-} \\ \text{ted components of } Gr \\ (\text{for } GL_n) \end{array} \right\} \xleftrightarrow{1:1} \mathbb{Z}_n$$

valuation (mod n)

<u>Slodowy slices.</u>

Let $\mathfrak{g}$ be a reductive Lie algebra and $x \in \mathcal{N} \subset \mathfrak{g}$ a nilpotent element.

Def-n. A transversal slice S_x in x to the (adjoint) orbit of x is a locally closed subvariety $S_x \subset \mathfrak{g}$, such that

- $x \in S_x$;
- the morphism $G \times S_x \to \mathfrak{g}$, $(g,s) \mapsto \mathrm{ad}(g)(s)$ is smooth;
- $\dim S_x = \mathrm{codim}(G \cdot x)$

In case $x \in \mathcal{N}$ such a slice is obtained as the affine space complimentary to the tangent space of the orbit $G \cdot x$ in $\mathfrak{g}$.

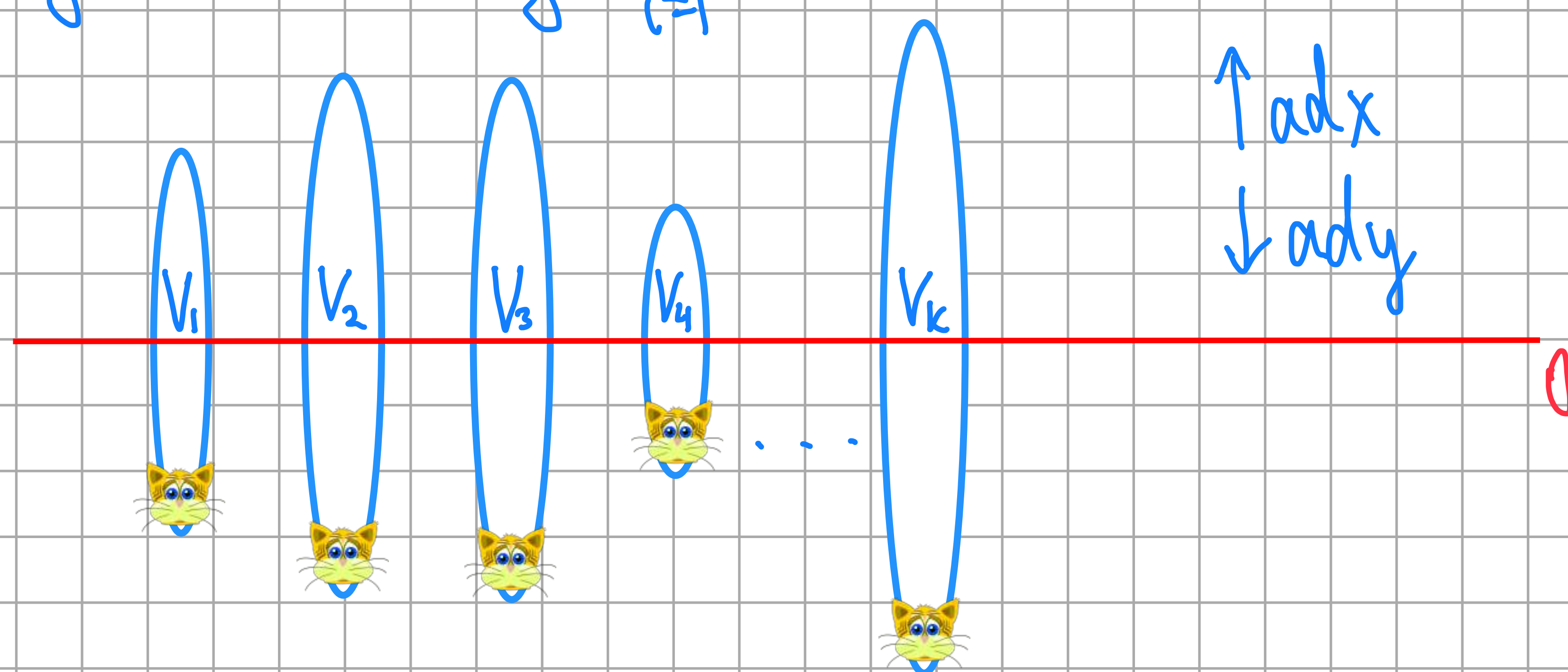
The recipe is as follows.

Step 1. We will need the Jacobson-Morozov theorem.

Thm. There exists a Lie algebra homomorphism $\varphi: \mathfrak{sl}_2 \to \mathfrak{g}$ with $\varphi\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = x$. All such homomorphisms are conjugate under the centralizer $Z_G(x)$.

The result above allows to complete x to an $\mathfrak{sl}_2$-triple $\langle x, y, h\rangle$, which will be denoted by $\mathfrak{g}^x$.

Step 2. Decompose $\mathfrak{g}$ into the sum of irreducible representations w.r.t. adjoint $\mathfrak{g}^x$-action: $\mathfrak{g} = \bigoplus_{i=1}^{k} V_i$



As $T_x(G\cdot x) = x + [\mathfrak{g}, x]$, the complement to $T_x(G\cdot x)$ in $\mathfrak{g}$ is $x +$  $= x + \ker(\mathrm{ad}\,y)$.
(consists of lowest weight vectors in V_i's).

Step 3. A slice to x inside $\mathcal{N}$ is $S_x \cap \mathcal{N}$.

Example. $\mathfrak{g} = \mathfrak{sl}_3$, $x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Step 1. As x is a positive root, $y=\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}$ is the corresponding negative root and

$$h=[x,y]=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}.$$

Step 2. Let $A\in\mathfrak{sl}_3$, then

$$[y,A]=\begin{pmatrix}-a_{13}&0&0\\-a_{23}&0&0\\a_{11}-a_{33}&a_{12}&a_{13}\end{pmatrix}$$

Thus, $A\in x+\ker(\operatorname{ad}y)$ is of the form

$$A=\begin{pmatrix}a&0&1\\b&-2a&0\\d&c&a\end{pmatrix}.$$

Step 3. Now we find the intersection

$S_x\cap\mathcal{N}=\left\{A=\begin{pmatrix}a&0&1\\b&-2a&0\\d&c&a\end{pmatrix}\middle|\ \chi_A(t)=t^3\right\}$, where $\chi_A(t)$ is the characteristic polynomial of A, i.e.

$\chi_A(t)=\det(A-t\cdot I)$.

The coefficient of t^2 is $\operatorname{tr}A=0$ $(A\in\mathfrak{sl}_3)$.

The coefficient of t is $2a^2+2a^2-a^2+d$.
The constant term is $\det(A)=-2a^3+2ad+bc$.
Hence, $S_x \cap \mathcal{N} = \mathbb{C}[a,b,c]/(bc-8a^3)$ is a kleinian singularity of type A_2 (the Dynkin diagram of $\mathfrak{sl}_3$).

We will need a little bit of preparation in order to formulate a more general result.

Def-n. An element $x \in \mathfrak{g}$ is called **regular** if its adjoint G-orbit is of maximal possible dimension. This is equivalent to $\dim Z_G(x) = \operatorname{rk} \mathfrak{g}$ (here $Z_G(x)$ is the centralizer of x).

An element $x \in \mathfrak{g}$ is called **subregular** if $\dim Z_G(x) = \operatorname{rk}\mathfrak{g}+2$.

Example. Let $\mathfrak{g} = \mathfrak{sl}_n$, $x = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \in \mathcal{N}$ and

$y = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \\ 0 & \cdots & & 0 \end{pmatrix} \in \mathcal{N}$. A direct calculation shows that

$$Z_G(x) = \left\{ \begin{pmatrix} 0 & a_1 & a_2 & \cdots & a_{n-1} \\ & 0 & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & a_2 \\ & & & \ddots & a_1 \\ 0 & & & & 0 \end{pmatrix} \right\}, \quad Z_G(y) = \left\{ \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & b \\ 0 & a_1 & a_2 \cdots & a_{n-2} & 0 \\ \vdots & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & & a_1 & 0 \\ 0 & 0 & \cdots 0 & c & (1-n)a_1 \end{pmatrix} \right\}.$$

As $\mathrm{rk}(\mathfrak{sl}_n)=n-1$, $\dim Z_G(x)=n-1$ and $\dim Z_G(y)=n+1$, x is regular and y is subregular.

<u>Thm (Dynkin).</u> If $\mathfrak{g}$ is simple, all subregular elements belong to the same conjugacy class.

Now we can state an interesting result.

<u>Thm (Brieskorn).</u> Let $\mathfrak{g}$ be a simple Lie algebra of type A, D or E and $x \in \mathcal{N} < \mathfrak{g}$ a subregular element. Then the variety $S_x \cap \mathcal{N}$ is a Kleinian singularity of type 'prescribed' by the Dynkin diagram of $\mathfrak{g}$.

Next we will show how the Slodowy slices (inside the nilcone) are realized in the affine Grassmannian.

<u>Birkhoff decomposition.</u>

Apart from the Cartan decomposition

$$G(K)=\bigsqcup_{\lambda \in \text{dom. coweights}} G(\mathcal{O})\, t^{\lambda}\, G(\mathcal{O})$$

there is the Birkhoff's decomposition:

$$G(K)=\coprod_{\substack{\lambda\in \text{dom.}\\ \text{coweights}}} G[t^{-1}]\,t^{\lambda}\,G(\mathcal{O})$$

The existence of this decomposition is equivalent to Grothendieck's thm classifying locally free sheaves (vector bundles) on the projective line $\mathbb{P}^1$.

<u>Thm (Grothendieck)</u>. Let E be a rank n locally free sheaf on $\mathbb{P}^1$, then $E\simeq\bigoplus_{i=1}^{n}\mathcal{O}(s_i)$, $s_i\in\mathbb{Z}$.

Recall that the line bundle $\mathcal{O}(k)$ on $\mathbb{P}^1$ is given by two modules $M_0\simeq\mathbb{C}[t]$ and $M_1\simeq\mathbb{C}[t^{-1}]$ (on the two affine charts $\mathbb{A}^1$) and transition function being multiplication by t^k.

Similarly, a rank n locally free sheaf on $\mathbb{P}^1$ is given by two modules $M_0\simeq\mathbb{C}[t]^n$ and $M_1\simeq\mathbb{C}[t^{-1}]^n$ (over $\mathbb{C}[t]$ and $\mathbb{C}[t^{-1}]$, respectively) together with a transition matrix $g\in GL_n(K)$. Notice that g is defined up to the change of basis in M_0 and M_1, i.e. action of $G[t^{-1}]$ on the left and $G[t]$ on the right. It follows that the Birkhoff's decomposition and Grothendieck's thm are equivalent.

Rmk. The attentive reader may have noticed that in Birkhoff's decomposition we act by $G(\mathcal{O})$ (the matrix entries are power series), while $G[t]$ above stands for matrices of polynomials, so instead of the decomposition above we rather need

$$G[t,t^{-1}] = \bigsqcup_{\substack{\lambda \in \text{dom.} \\ \text{coweights}}} G[t^{-1}]\, t^{\lambda}\, G[t],$$

which also holds true and bears Birkhoff's name.

Slices in affine Grassmannian.

Let $Gr^{\mu} := G[t^{-1}] \cdot t^{\mu} \subset Gr$.

Thm. (1) $Gr^{\mu} \cap Gr_{\lambda} = \emptyset$ if $\mu > \lambda$.

(2) $Gr^{\mu} \cap Gr_{\mu} \simeq G \cdot t^{\mu}$

Rmk. The proof is a straightforward calculation. The variety $G\cdot t^\mu$ is the fixed point set for the action of one-dimensional torus $\mathbb{C}^\times$ on Gr_μ via rescaling t. This torus is called the rotation torus.

Let $G_1 \subset G[t^{-1}]$ be the kernel of the evaluation map $\varphi: G[t^{-1}] \longrightarrow G$

$$t^{-1} \longmapsto 0$$

and $\widetilde{Gr^\mu} := G_1 \cdot t^\mu$.

Then $\widetilde{Gr^\mu} \cap Gr_\mu = t^\mu$ is a single point.

Prop-n. Let $\mu \preceq \lambda$, then $\widetilde{Gr^\mu} \cap \overline{Gr_\lambda}$ intersects Gr_ν transversally for any $\mu \preceq \nu \preceq \lambda$.

In particular, for $\lambda = (n, 0, 0, \dots, 0)$ and $\mu \preceq \lambda$, one gets $\widetilde{Gr^\mu} \cap \overline{Gr_\lambda} \simeq S_x \cap \mathcal{N}$, where the Jordan form of the nilpotent matrix x has partition type μ.