Nakajima quiver varieties & affine Grassmannians

Motivating example

Let $\mathbf{v} = (v_1, v_2, \ldots, v_s) \in \mathbb{Z}^s_{\geq 0}$ with $v_1 \leq v_2 \leq \ldots \leq v_s$ be an s-tuple of nondecreasing positive integers and

$$\mathcal{F}\ell(\mathbf{v}) = \{V_1 \subseteq V_2 \subseteq \ldots \subseteq V_s \mid \dim(V_j) = v_j\}$$

the corresponding s-step variety of partial flags. We will denote the standard basic vectors by $\varepsilon_i = (0, \ldots, 0, \underset{i}{1}, 0, \ldots, 0)$ and fix the dimension of the ambient space to be n.

Definition. *Let $\mathbf{v}' = (v_1, \ldots, v_i, v_i + 1, v_{i+1}, \ldots v_s) \in \mathbb{Z}^{s+1}_{\geq 0}$. We will call the $(s+1)$-step varieties*

$$\mathcal{Z}_{\mathbf{v},\mathbf{v}+\varepsilon_i} = \mathcal{F}\ell(\mathbf{v}')$$

the ***incidence varieties****.*

Remark. *There is a natural embedding $\mathcal{Z}_{\mathbf{v},\mathbf{v}+\varepsilon_i} \subset \mathcal{F}\ell(\mathbf{v}) \times \mathcal{F}\ell(\mathbf{v}+\varepsilon_i)$*

Next we consider the projections $p : \mathcal{Z}_{\mathbf{v},\mathbf{v}+\varepsilon_i} \to \mathcal{F}\ell(\mathbf{v})$ and $q : \mathcal{Z}_{i|i+1} \to \mathcal{F}\ell(\mathbf{v}+\varepsilon_i)$. The first map ($p$) simply 'forgets' the $i+1$st subspace and the second map (q) forgets the ith subspace.

$$\begin{tikzcd} & \mathcal{Z}_{\mathbf{v},\mathbf{v}+\varepsilon_i} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathcal{F}\ell(\mathbf{v}) & & \mathcal{F}\ell(\mathbf{v}+\varepsilon_i) \end{tikzcd}$$

Remark. *We have that both p and q are projective bundles with respective fibers being $\mathbb{P}\left(V_{i+1}/V_i\right)$ and $\mathbb{P}\left(\left(\tilde{V}_i/\tilde{V}_{i-1}\right)^*\right)$.*

Let $L_{\mathbf{v}}$ be the space of constant functions on $\mathcal{F}\ell(\mathbf{v})$, i.e.

$$L_{\mathbf{v}} = \begin{cases} \mathbb{C}, & \mathcal{F}\ell(\mathbf{v}) \neq \varnothing \\ \{0\}, & \mathcal{F}\ell(\mathbf{v}) = \varnothing. \end{cases}$$

Similarly, $L_{\mathcal{Z}_{\mathbf{v},\mathbf{v}+\varepsilon_i}} = L_{\mathbf{v}'}$.

Our next goal is to define the action of Lie algebra $\mathfrak{sl}_{s+1}$ (or $\mathfrak{gl}_{s+1}$) on the collection of spaces $\{L_{\mathbf{v}}\}_{\mathbf{v}\in\mathbb{Z}^s_{\geq 0}}$.

Remark. *Notice that dim* $\left(\bigoplus_{\mathbf{v}} L_{\mathbf{v}}\right) = dim(\mathbb{C}[x_0, x_1, \ldots, x_s]_n) = \binom{n+s}{n}$ *(the differences of dimensions of subsequent subspaces sum up to the dimension of the ambient space equal to* n*). The latter (space of homogeneous polynomials of degree* n *in* $s+1$ *variables) is the representation we hope to (and will) get.*

Define the pullback and pushforward via

$$\begin{aligned}
p^* &: L_{\mathbf{v}} \to L_{\mathbf{v}'}, 1 \mapsto 1 \\
q^* &: L_{\mathbf{v}} \to L_{\mathbf{v}'}, 1 \mapsto 1 \\
p_! &: L_{\mathbf{v}'} \to L_{\mathbf{v}}, 1 \mapsto \chi(F_p) = v_{i+1} - v_i - 1 \\
q_! &: L_{\mathbf{v}'} \to L_{\mathbf{v}}, 1 \mapsto \chi(F_q) = v_i - v_{i-1}.
\end{aligned}$$

Here F_p and F_q stand for respective fibers over a (any) point and $\chi(F_p), \chi(F_q)$ are the Euler characteristics.

Define

$$\begin{aligned} e_{i,\mathbf{v}+\varepsilon_i} &:= q_! p^*, \\ f_{i,\mathbf{v}} &:= p_! q^*, \\ e_i &:= \bigoplus_{\mathbf{v}} e_{i,\mathbf{v}}, \\ f_{i,\mathbf{v}} &:= \bigoplus_{\mathbf{v}} f_{i,\mathbf{v}}. \end{aligned}$$

It remains to define the action of h_i via $[e_i, f_i]$ and is direct to check that all required relations between those are satisfied.

Nakajima quiver varieties

Let $Q = (Q_0, Q_1)$ be a finite quiver, i.e. a directed graph with finitely many vertices enumerated by the set Q_0 and finitely many edges enumerated by Q_1. Each edge is uniquely determined by the pair of vertices it connects, which we will denote by $t(a)$ and $h(a)$ standing for 'tail' and 'head'. Consider two dimension vectors $\mathbf{v} = (v_1, \dots, v_n)$ and $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{Z}^n_{\geq 0}$, where n is the cardinality of Q_0 and form a vector space

$$R = \bigoplus_{a \in Q_1} \mathrm{Hom}_{\mathbb{C}}(V_{ta}, V_{ha}) \oplus \bigoplus_{s \in Q_0} \mathrm{Hom}_{\mathbb{C}}(V_s, W_s).$$

Remark. *The dimension vector* $\mathbf{w}$ *is often referred to as framing.*

Notice that the space T^*R is symplectic and naturally identified with

$$\bigoplus_{a\in Q_1}\left(\mathrm{Hom}_{\mathbb{C}}(V_{ta},V_{ha})\oplus \mathrm{Hom}_{\mathbb{C}}(V_{ha},V_{ta})\right)\oplus \bigoplus_{s\in Q_0}\left(\mathrm{Hom}_{\mathbb{C}}(V_s,W_s)\oplus \mathrm{Hom}_{\mathbb{C}}(W_s,V_s)\right).$$

We will use the notation $(\mathbf{x},\bar{\mathbf{x}},\mathbf{i},\mathbf{j})$ to represent a point $p\in T^*R$, where

$$\begin{aligned}
\mathbf{x} &= (x_a\in \mathrm{Hom}_{\mathbb{C}}(V_{ta},V_{ha}))_{a\in Q_1},\\
\bar{\mathbf{x}} &= (x_{a^*}\in \mathrm{Hom}_{\mathbb{C}}(V_{ha},V_{ta}))_{a\in Q_1},\\
\mathbf{i} &= (i_s\in \mathrm{Hom}_{\mathbb{C}}(V_s,W_s))_{s\in Q_0} \text{ and}\\
\mathbf{j} &= (j_s\in \mathrm{Hom}_{\mathbb{C}}(W_s,V_s))_{s\in Q_0}.
\end{aligned}$$

The reductive group $G := \prod_{i=1}^{n} GL(V_i)$ naturally acts on R. We are interested in the induced Hamiltonian action of G on T^*R. The corresponding moment map $\mu : T^*R \to \mathfrak{g}^*$ is given by

$$\mu(\mathbf{x}, \bar{\mathbf{x}}, \mathbf{i}, \mathbf{j}) = \sum_{a \in Q_1} (x_a x_{a^*} - x_{a^*} x_a) - \sum_{s \in Q_0} j_s i_s. \tag{1}$$

To define the Nakajima quiver variety $\mathcal{M}^\theta(Q, \mathbf{v}, \mathbf{w})$, we need to choose some character θ of G. Such θ is uniquely determined by an n-tuple of integers, i.e. by $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{Z}^n$ we understand the character θ as a map $(g_1, \ldots, g_n) \mapsto \prod_{s=1}^{n} \det(g_s)^{\theta_s}$, where $g_s \in GL(V_s)$.

Definition. *The GIT quotient* $\mathcal{M}_0^\theta(Q, \mathbf{v}, \mathbf{w}) := \mu^{-1}(0)^{\theta-ss}//^\theta G$ *is called the Nakajima quiver variety with parameter* θ*.*

Agreement. *Henceforth, we assume that* Q *has no loops.*

The following results are due to Nakajima.

Theorem.

1. *The variety* $\mathcal{M}_0^0(Q, \mathbf{v}, \mathbf{w})$ *is affine and there is a projective morphism* $\rho : \mathcal{M}_0^\theta(Q, \mathbf{v}, \mathbf{w}) \to \mathcal{M}_0^0(Q, \mathbf{v}, \mathbf{w})$*, which is a symplectic resolution of singularities for generic* θ*.*
2. *Every irreducible component of the central fiber* $\Lambda(\mathbf{v}, \mathbf{w}) := \rho^{-1}(0) \subset \mathcal{M}_0^\theta(Q, \mathbf{v}, \mathbf{w})$ *is a (locally closed) Lagrangian subvariety.*

Remark. *An application of the Hilbert-Mumford criterion shows that :*

1. *if* $\theta_t > 0 \ \forall t$, *then a quadruple* $(\mathbf{x}, \bar{\mathbf{x}}, \mathbf{i}, \mathbf{j}) \in \mu^{-1}(0)$ *is* θ*-semistable if and only if for any collection of vector subspaces* $S = (S_t)_{t \in Q_0} \subset V = (V_t)_{t \in Q_0}$*, which is stable under the maps* $\mathbf{x}, \bar{\mathbf{x}}$*, we have*

$$S_t \subset \ker(i_t) \ \forall t \in Q_0 \Rightarrow S = 0,$$

2. *if* $\theta_t < 0 \ \forall t$, *then a quadruple* $(\mathbf{x}, \bar{\mathbf{x}}, \mathbf{i}, \mathbf{j}) \in \mu^{-1}(0)$ *is* θ*-semistable if and only if for any collection of vector subspaces* $S = (S_t)_{t \in Q_0} \subseteq V = (V_t)_{t \in Q_0}$*, which is stable under the maps* $\mathbf{x}, \bar{\mathbf{x}}$*, we have*

$$S_t \supset im(j_t) \ \forall t \in Q_0 \Rightarrow S = V.$$

Next we describe quiver varieties associated to some quivers.

Example. $Q = \bullet$.

First consider the quiver with one vertex and no arrows. In this case $R = Hom_{\mathbb{C}}(V, W), G = GL(V)$ *and* $T^*R = Hom_{\mathbb{C}}(V, W) \oplus Hom_{\mathbb{C}}(W, V)$*. The moment map is* $\mu(i, j) = -ji$.

$$\begin{tikzcd} W \arrow[d, shift left, "j"] \\ V \arrow[u, shift left, "i"] \end{tikzcd}$$

Quiver variety for $Q = \bullet$.

Let $\theta > 0$, then due to the Remark we made before, $\mu^{-1}(0)^{\theta-ss}$ is formed by $\{i, j \mid ji = 0\}$ with j injective. The choice of such a pair (i, j) is equivalent to a choice of a subspace $V \subset W$ and a map in $\mathrm{Hom}(W/V, V)$, which is naturally an element of the cotangent bundle $T^*Gr(v, w)$. We conclude that $\mathcal{M}_0^{\theta}(Q, v, w) \simeq T^*Gr(v, w)$.

In case $\theta < 0$ our Remark before asserts the surjectivity of j. Since $ji = 0$, the image of i must be contained in the kernel of j, and the latter is isomorphic to W/V as j is surjective. So j is uniquely determined by its kernel, a $(w - v)$-dimensional subspace of W, while $i \in \mathrm{Hom}(V, W/V)$. This allows to identify $\mathcal{M}_0^{\theta}(Q, v, w)$ with $T^*Gr(w - v, w) \simeq T^*Gr(v, w)$.

It remains to see what happens when $\theta = 0$. Notice that the product ij is G-invariant and $(ij)^2 = 0$, since ji vanishes. Construct the map $\phi : M_0^0(Q, v, w) \to \mathfrak{gl}(W)$ by setting $\phi(i, j) = ij$. Furthermore, the ring of invariants $\mathbb{C}[i, j]^G$ is generated by the matrix elements of the product ij. Writing $W = ker\ j \oplus W'$ and observing that $ji = 0$ implies $im\ ij \subset ker\ j$, we conclude that $rk(ij) \leq min\left(v, \lfloor \frac{w}{2} \rfloor\right)$. Denote the number $min\left(v, \lfloor \frac{w}{2} \rfloor\right)$ by k, then the image of ϕ consists of $A \in \mathfrak{gl}(W)$, s.t.

1. $A^2 = 0$ and
2. $rk(A) \leq k$.

Consider a nilpotent matrix x in $\mathfrak{gl}(W)$, whose Jordan canonical form consists of k blocks of size 2×2 and $w - 2k$ blocks of size 1×1. Matrices A in the image of ϕ are in $\overline{\mathcal{O}}_x$, the closure of the orbit of x. We find that $M_0^0(Q, v, w)$ is isomorphic to $\overline{\mathcal{O}}_x$.

Example. *We generalize the previous example and choose a Dynkin quiver of type A_s. Take an arbitrary dimension vector v and w with $w_1 = \ldots = w_{s-1} = 0$ and a stability condition $\theta = (\theta_1, \ldots, \theta_s)$ with all $\theta_t > 0$.*

$$\begin{tikzcd}
W \arrow[d, shift left, "j"] & & & & & \\
V_s \arrow[u, shift left, "i"] \arrow[r, shift left, "x_{ss-1}"] & V_{s-1} \arrow[l, shift left, "y_{s-1s}"] \arrow[r, shift left, "x_{s-1s-2}"] & V_{s-2} \arrow[l, shift left, "y_{s-1s-2}"] & \cdots & V_2 \arrow[r, shift left, "x_{21}"] & V_1 \arrow[l, shift left, "y_{12}"]
\end{tikzcd}$$

Quiver variety for $Q = A_s$.

Then $M_0^\theta(Q, v, w)$ is the cotangent bundle of the partial flag variety $\mathcal{F}\ell(v_1, \ldots, v_s; w)$ (or empty if $v_i > v_{i+1}$ for some i or $v_s > w_s$). On the other hand $M_0^0(Q, v, w) \simeq \overline{\mathcal{O}}_x$, where $x \in \mathfrak{gl}(W)$ is a nilpotent element, having blocks of sizes $v_1, v_2 - v_1, \ldots, v_s - v_{s-1}, w - v_s$ in its Jordan canonical form. The map

$$\rho : M_0^\theta(Q, v, w) = T^*(G/P) \to \overline{\mathcal{O}}_x = M_0^0(Q, \mathbf{v}, \mathbf{w})$$

is the Springer resolution.

Constructible functions

Let X be an algebraic variety and $\mathrm{Fun}(X, \mathbb{C})$ the algebra of all possible functions from X to $\mathbb{C}$.

Example. *If $Y \subset X$ is a locally closed subvariety (in Zariski topology) define*

$$\delta_Y(x) := \begin{cases} 1, x \in Y \\ 0, x \notin Y. \end{cases}$$

Definition. *The subalgebra $Fun_c(X, \mathbb{C}) = Span_{\mathbb{C}}\{\delta_Y \mid Y \subset X, \text{ loc. closed}\}$ is called the **subalgebra of constructible functions**.*

Remark. *If $f \in Fun_c(X, \mathbb{C})$, then $f^{-1}(a)$ is a constructible subset (finite union of loc. closed sets) for any $a \in \mathbb{C}$.*

Definitions/Properties. *Let $\varphi : X \to Y$ be a morphism.*

1. *If $f \in Fun_c(Y, \mathbb{C})$, then $\varphi^*(f)$ given by $\varphi^*(f)(x) := f(\varphi(x))$ is in $f \in Fun_c(X, \mathbb{C})$.*

2. *If $g \in Fun_c(X, \mathbb{C})$, then $\varphi_!(g)$ given by $\varphi_!(g)(y) := \sum_{a \in \mathbb{C}} \chi_c(\varphi^{-1}(y) \cap f^{-1}(a))a$ is in $f \in Fun_c(Y, \mathbb{C})$.*

Remark. *Each $\varphi^{-1}(y) \cap f^{-1}(a)$ is a constructible set, χ_c stands for Euler characteristic with compact support and $f^{-1}(a) \neq \varnothing$ only for finitely many $a \in \mathbb{C}$.*

Constructing representations

Starting with a quiver $Q = (Q_0, Q_1)$, one constructs a matrix $C = (c_{ij})_{i,j \in Q_0}$ with $c_{ii} := 2$ and $c_{ij} := -\#\{\text{arrows between } i \text{ and } j\}$, which (considered as a Cartan matrix) gives rise to a Lie algebra $\mathfrak{g}_Q$.

Analogously to the construction we started with, let

$$\begin{aligned} L^{\mathbf{w}}_{\mathbf{v}} &:= \mathrm{Fun}_c(\Lambda(\mathbf{v}, \mathbf{w}), \mathbb{C}) \\ L^{\mathbf{w}}_{\mathbf{v}+\varepsilon_i} &:= \mathrm{Fun}_c(\Lambda(\mathbf{v}+\varepsilon_i, \mathbf{w}), \mathbb{C}) \\ L^{\mathbf{w}}_{\mathbf{v},\mathbf{v}+\varepsilon_i} &:= \mathrm{Fun}_c(\Lambda(\mathbf{v}, \mathbf{v}+\varepsilon_i, \mathbf{w}), \mathbb{C}). \end{aligned}$$

Using the diagram below, define e_i and f_i via

$$\begin{aligned} e_{i,\mathbf{v}+\varepsilon_i} &:= q_! p^*, \\ f_{i,\mathbf{v}} &:= p_! q^*, \\ e_i &:= \bigoplus_{\mathbf{v}} e_{i,\mathbf{v}}, \\ f_{i,\mathbf{v}} &:= \bigoplus_{\mathbf{v}} f_{i,\mathbf{v}}. \end{aligned}$$

$$\begin{tikzcd} & \Lambda(\mathbf{v}, \mathbf{v}+\varepsilon_i, \mathbf{w}) \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \Lambda(\mathbf{v}, \mathbf{w}) & & \Lambda(\mathbf{v}+\varepsilon_i, \mathbf{w}) \end{tikzcd}$$

Let $\{\pi_i\}_{i \in Q_0}$ be the fundamental weights, i.e. $\langle \pi_i, h_j \rangle = \delta_{ij}$ and $\{\alpha_i\}_{i \in Q_0}$ simple roots.

Theorem. *The operators e_i, f_i and $h_i := [e_i, f_i]$ with $i \in Q_0$ make the space $L^{\mathbf{w}} := \bigoplus_{\mathbf{v}} L^{\mathbf{w}}_{\mathbf{v}}$ into an integrable $\mathfrak{g}_Q$ representation. Moreover, this representation is the one with highest weight $w = \sum_{i \in Q_0} w_i \pi_i$, the corresponding one-dimensional subspace of weight w is $L^{\mathbf{w}}_0$ (notice that $\mathcal{M}^{\theta}_0(Q, 0, \mathbf{w})$ is a point, hence, $L^{\mathbf{w}}_0 = \mathbb{C}$), more generally, $L^{\mathbf{w}}_{\mathbf{v}}$ is the weight subspace of weight $w - \sum_{i \in Q_0} v_i \alpha_i$.*

Variation: Borel-Moore homology

Definition. *Let M be a smooth oriented manifold. The* ***Borel-Moore homology*** *of a closed subset $X \subset M$ is the relative homology*

$$H_{BM} := H_\bullet(M, M \setminus X, \mathbb{C}).$$

Remark. *If X is an irreducible complex algebraic variety, the space $H_{top}(X)$ is 1-dimensional. In the ordinary homology theory, fundamental classes only exist for compact manifolds, while the fundamental class exists in Borel-Moore homology for noncompact X as well.*

Four years after establishing the Theorem that we saw on the previous slide, Nakajima found the following 'reformulation'.

Theorem. $L^{\mathbf{w}} \simeq \bigoplus_{\mathbf{v}} H_{BM}^{top}(\Lambda(\mathbf{v}, \mathbf{w}))$.

Connection to affine Grassmannians

Definition. *Let $G_1 \subset G[t^{-1}]$ be the kernel of the evaluation map $G[t^{-1}] \to G$ and $\mu \prec \lambda$ two dominant coroots. The intersection $Gr^{\mu}_{\lambda} := G_1 \cdot t^{\mu} \cap \overline{Gr_{\lambda}}$ is called the* ***Luzstig slice****.*

Theorem. *Let $\mathcal{O}_{\mu}, \mathcal{O}_{\lambda} \subset \mathcal{N} \subset \mathfrak{gl}_N$, T_{μ} be the Mirkovic-Vybornov transverse slice to the nilpotent orbit $\mathcal{O}_{\mu}$ in the nilpotent cone $\mathcal{N}$ (the slice T_{μ} different from the Slodowy slice), $\eta : \widetilde{\mathcal{O}}_{\mu} \to \overline{\mathcal{O}}_{\mu}$ the Springer resolution and $\pi : \widetilde{Gr}_{\lambda} \to \overline{Gr}_{\lambda}$ the 'convolution' resolution constructed by Mirkovic and Vilonen. The following diagram commutes and the horizontal arrows are isomorphisms.*

$$\begin{tikzcd}
\mathcal{M}_0^{\theta}(Q, \mathbf{v}, \mathbf{w}) \arrow[r, leftrightarrow, "\simeq"] \arrow[d, "\rho"'] & \eta^{-1}(T_{\mu} \cap \overline{\mathcal{O}}_{\lambda}) \arrow[r, leftrightarrow, "\simeq"] \arrow[d, "\eta"'] & \pi^{-1}(Gr^{\mu}_{\lambda}) \arrow[d, "\pi"'] \\
\mathcal{M}_0(Q, \mathbf{v}, \mathbf{w}) \arrow[r, leftrightarrow, "\simeq"] & T_{\mu} \cap \overline{\mathcal{O}}_{\lambda} \arrow[r, leftrightarrow, "\simeq"] & Gr^{\mu}_{\lambda}
\end{tikzcd}$$

Remark. *We sketch the construction of isomorphisms in the right square of the commutative diagram on the previous slide. Let* $\lambda = (\lambda_1, \ldots, \lambda_k)$ *and* $\widehat{\lambda} = (a_1, \ldots, a_s)$ *be the dual partition. The preimage of a lattice* $\Lambda \in \overline{Gr}_\lambda$ *in* $\widetilde{Gr}_\lambda$ *consists of partial flags of lattices*

$$\pi^{-1}(\Lambda) = \{\Lambda_0 \subseteq \Lambda_1 \subseteq \ldots \subseteq \Lambda_s = \Lambda \mid dim_{\mathbb{C}}(\Lambda_i/\Lambda_{i-1}) = a_i, t(\Lambda_i) \subset \Lambda_{i-1}\}.$$

Recall that the Springer resolution of the closure of nilpotent orbit $\mathcal{O}_x$*, where* x *has a Jordan normal form of type* λ *consists of the pairs* $(y, \mathcal{F})$ *with* $y \in \overline{\mathcal{O}}_x$ *and* $\mathcal{F}$ *an* s*-step partial flag stabilized by* y *with relative dimensions equal to corresponding elements of* $\widehat{\lambda}$*. This gives*

$$\nu^{-1}(y) = \{F \in \mathcal{F}\ell(\widehat{\lambda}) \mid yF_i \subset F_{i-1}\}.$$

It is straightforward to identify $y \in \overline{\mathcal{O}}_x$ *with* $\Lambda \in \overline{Gr}_\lambda$ *via* y *being the matrix for the* t*-action on* Λ/Λ_0*. Moreover, an element in* $\pi^{-1}(\Lambda)$ *(as a partial flag in* Λ/Λ_0 *of the required 'shape') corresponds to an element in* $\nu^{-1}(y)$*.*