Mirkovic-Vilonen cycles and polytopes

Satake isomorphism

Let $\mathcal{O} = \mathbb{C}[|t|]$ be a ring of formal power series, G be a connected, reductive algebraic group over $\mathcal{K} = \mathrm{Frac}(\mathcal{O})$ and K a maximal compact subgroup (for instance, $K = G(\mathcal{O})$).

The Hecke ring $\mathcal{H} = \mathcal{H}(G, K)$ is by definition the ring of all locally constant, compactly supported functions $f : G \to \mathbb{Z}$ which are K-biinvariant:

$$f(kx) = f(xk') = f(x)$$

for all $k, k' \in K$. The multiplication in $\mathcal{H}$ is via convolution

$$f \star g(z) = \int_G f(x)g(x^{-1}z)dx$$

where dx is the unique Haar measure on G, s.t. K has volume 1.

Remark. *Each function* $f \in \mathcal{H}$ *is constant on double cosets* KxK, *since it is also compactly supported, it is a finite linear combination of the characteristic functions* $char(KxK)$ *of double cosets. Hence these characteristic functions give a* $\mathbb{Z}$*-basis for* $\mathcal{H}$.

Theorem. *There is an isomorphism of rings*

$$(\mathcal{H}, \star) \simeq (Rep(G^\vee) \otimes \mathbb{C}, \otimes).$$

Example. *Let* $G = \mathbb{C}^*$. *The Cartan decomposition gives*

$$G(\mathcal{K}) = \mathcal{K}^* = \bigsqcup_{m \in \mathbb{Z}} \mathcal{O}^* t^m \mathcal{O}^*$$

with $Gr_G \simeq \mathbb{Z} = \bigcup_{n \geq 0} ([-n, n] \cap \mathbb{Z})$ *and* $K = \mathcal{O}^*$.

We have $\mathcal{H}(G(\mathcal{K}), K)) = \mathcal{H}(\mathcal{K}^*, \mathcal{O}) = \mathrm{Fun}^c_{\mathcal{O}^* \times \mathcal{O}^*}(\mathcal{K}^*, \mathbb{C}) = \mathrm{Fun}^c(\mathbb{Z}, \mathbb{C})$. Next notice that $G^\vee = G = \mathbb{C}^*$ and there is a ring isomorphism

$$\varphi : \mathrm{Fun}^c(\mathbb{Z}, \mathbb{C}) \to (\mathrm{Rep}(\mathbb{C}^*) \otimes \mathbb{C}, \otimes).$$

If the range of $\psi \in \mathrm{Fun}^c(\mathbb{Z}, \mathbb{C})$ is a subset of $\mathbb{Z}_{\geq 0}$, then $\varphi(\psi) = \bigotimes_{i \in \mathbb{Z}} V_i^{\oplus \psi(i)}$, where V_i is the one-dimensional representation of $\mathbb{C}^*$ (the action of $1 \in \mathbb{C}^*$ on V_i is via ith primitive root of unity). For instance, let $\chi_i \in \mathcal{H}$ be the characteristic function of $i \in \mathbb{Z}$, i.e. $\chi_i(k) = \delta_{i,k}$. Hence

$$\chi_i \star \chi_j(a) = \sum_{s \in \mathbb{Z}} \chi_i(s)\chi_j(a-s) = \delta_{i+j,a} = \chi_{i+j}(a).$$

On the other hand, $V_i \otimes V_j = V_{i+j}$, so $\varphi(\chi_i \star \chi_j) = \varphi(\chi_{i+j}) = \varphi(\chi_i) \otimes \varphi(\chi_j)$.

Geometric Satake isomorphism

There is an isomorphism of tensor categories

$$\mathrm{Perv}_{G(\mathcal{O})}(Gr_G, \Bbbk) \simeq \mathrm{Rep}(G^\vee),$$

where $\Bbbk$ is a Noetherian commutative ring with unit and of finite global dimension ($\Bbbk = \mathbb{C}, \mathbb{Z}, \overline{\mathbb{F}}_q, \ldots$)

Recall that for a smooth manifold M of dimension n, the cohomology of M satisfies the Poincare duality, i.e.

$$H^i(M, \mathbb{C}) \simeq H^{n-i}(M, \mathbb{C}).$$

Moreover, there is a 'sheaf way' to get the cohomology of M. Let $\underline{\mathbb{C}}_M$ be the sheaf of locally constant functions on M: for any open connected $U \subset M$ one has $\underline{\mathbb{C}}_M(U) = \mathbb{C}$. Then there is an isomorphism of graded algebras

$$H^*(M, \underline{\mathbb{C}}_M) \simeq H^*(M, \mathbb{C}).$$

Question. *What if the manifold X is not smooth?*

In case X admits a 'good enough' stratification, Goresky and Macpherson found the 'right' version of homology that satisfies Poincare duality. They called it intersection homology and following a request of Deligne 'sheafified' it to get IC sheaves. Analogously to the case of smooth manifolds, there is an isomorphism

$$IH^*(X) \simeq \mathbb{H}^*(X, IC(X)).$$

The sheaves $\mathrm{IC}(\overline{X}_\lambda)$ for affine Schubert cells $X_\lambda = G(\mathcal{O})t^\lambda$ play a fundamental role in the geometric Satake correspondence, namely, $\mathrm{IC}(\overline{X}_\lambda)$ corresponds to the irreducible highest weight representation of $G^\vee$ given by the coweight λ.

Mirkovic-Vilonen cycles

Fix $T \subset B \subset G$ and let $N \subset B$ be the unipotent radical with $N(\mathcal{K}) \subset G(\mathcal{K})$.
If $G = GL_n$ and B consists of upper-triangular matrices, then

$$N(\mathcal{K}) = \begin{pmatrix} 1 & * & \dots & * \\ 0 & 1 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

with each $*$ being an element of $\mathcal{K}$. Let $\mathcal{S}_\lambda := N(\mathcal{K})t^\lambda$.

Remark. *$\mathcal{S}_\lambda$ is neither of finite dimension nor of finite codimension in Gr_G.*

Theorem. *The intersection $\mathcal{S}_\nu \cap X_\lambda \neq \varnothing$ if and only if $t^\nu \in \overline{X}_\lambda$, in which case $dim(\mathcal{S}_\nu \cap X_\lambda) = \rho(\nu + \lambda)$.*

For any $n \in N(\mathcal{K})$ we have $\lim_{s \to 0} 2\overset{\vee}{\rho}(s)n = I$ (the action is via conjugation and all elements above diagonal have positive s-weights). It follows that $\mathcal{S}_\nu$ can be alternatively defined as

$$\mathcal{S}_\nu = \{x \in Gr_G \mid \lim_{s \to 0} 2\overset{\vee}{\rho}(s)x = t^\nu\}.$$

Theorem. *For each λ we have a decomposition*

$$IH_*(\overline{X}_\lambda) = \bigoplus_{\nu \preceq \lambda} H_{top}(\overline{X}_\lambda \cap \mathcal{S}_\nu).$$

Definition. *The* ***Mirkovic-Vilonen cycles*** *are the irreducible components of $\overline{X_\lambda \cap \mathcal{S}_\nu}$.*

Theorem. *Mirkovic-Vilonen cycles give a basis of $H_{top}(\overline{X}_\lambda \cap \mathcal{S}_\nu)$.*

Example. *Consider* $G = GL_n$ *and the minuscule weight* $\lambda_k = (\underbrace{1,\ldots,1}_{k},\underbrace{0,\ldots,0}_{n-k})$.

Recall that $\overline{X}_{\lambda_k} = X_{\lambda_k} \simeq Gr(n-k,n)$ *and there is the Plucker embedding*

$$\mathcal{P} : Gr(n-k,n) \hookrightarrow \mathbb{P}^{\binom{n}{n-k}-1},$$

given by $\mathcal{P}(W) = w_1 \wedge w_2 \wedge \ldots \wedge w_{n-k}$ *for any* $W = span(w_1, w_2, \ldots, w_{n-k}) \in Gr(n-k,n)$. *Next we find the fixed points for the action of the one-parameter subgroup* $\lambda : \mathbb{C}^* \to T$ *with* $\lambda(s) = diag(s^{n-1}, s^{n-2}, \ldots, s, 1)$ *(this subgroup contracts* $N(\mathcal{K})$ *to a point and can be used instead of* $2\overset{\vee}{\rho}$*). This group naturally acts on* V *giving rise to an action on* $\Lambda^{n-k}(V)$ *and, hence, on* $\mathbb{P}^{\binom{n}{n-k}-1} = \mathbb{P}(\Lambda^{n-k}(V))$. *There are* $\binom{n}{n-k}$ *fixed points corresponding to the 'coordinate wedges'* $e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_{n-k}}$ *with* $1 \leq i_1 < i_2 < \ldots < i_{n-k} \leq n$. *Such a fixed point, in turn, corresponds to the point*

$$diag(1,\ldots,1,\underbrace{t}_{i_1},1,\ldots,1,\underbrace{t}_{i_2},1,\ldots,\ldots,1,\underbrace{t}_{i_{n-k}},1,\ldots,1) \in X_{\lambda_k} \cap \mathcal{S}_{\nu_{i_1 i_2 \ldots i_{n-k}}}$$

with $\nu_{i_1 i_2 \ldots i_{n-k}} = (0,\ldots,0,\underbrace{1}_{i_1},0,\ldots,0,\underbrace{1}_{i_2},0,\ldots,\ldots,0,\underbrace{1}_{i_{n-k}},0,\ldots,0) \prec \lambda_k.$

As one of the descriptions of the Schubert cells (in the usual Grassmannian) is via attracting loci w.r.t. the one-parameter subgroup λ-action, we conclude that the MV cycles $X_\lambda \cap \mathcal{S}_{\nu_{i_1 i_2 \ldots i_{n-k}}}$ are exactly the Schubert cells. Moreover, there is a natural one-to-one correspondence between these cells and the basis of $\Lambda^{n-k}(V)$, an irreducible representation of GL_n, via

$$(X_{\lambda_k} \cap \mathcal{S}_{\nu_{i_1 i_2 \ldots i_{n-k}}}) \leftrightarrow e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_{n-k}}.$$

One more example.

Example. *Let* $G = SL_2$ *and consider the dominant weights* $\lambda = m$ *and* $\nu_k = k \leq m$. *For* $n = \begin{pmatrix} 1 & f(t,t^{-1}) \\ 0 & 1 \end{pmatrix} \in N(\mathcal{K})$ *we compute* $n \cdot \begin{pmatrix} t^k & 0 \\ 0 & t^{-k} \end{pmatrix} = \begin{pmatrix} t^k & t^{-k} f(t,t^{-1}) \\ 0 & t^{-k} \end{pmatrix}$, *while* $\begin{pmatrix} g_{11}(t) & g_{12}(t) \\ g_{21}(t) & g_{22}(t) \end{pmatrix} \cdot \begin{pmatrix} t^m & 0 \\ 0 & t^{-m} \end{pmatrix} = \begin{pmatrix} t^m g_{11}(t) & t^{-m} g_{12}(t) \\ t^m g_{21}(t) & t^{-m} g_{22}(t) \end{pmatrix}$ *for* $g = \begin{pmatrix} g_{11}(t) & g_{12}(t) \\ g_{21}(t) & g_{22}(t) \end{pmatrix} \in SL_2(\mathcal{O})$. *It follows that (up to right* $SL_2(\mathcal{O})$*-action)*

$$\overline{X}_\lambda \cap \mathcal{S}_{\nu_k} = \left\{ \begin{pmatrix} t^k & t^{-k} f(t,t^{-1}) \\ 0 & t^{-k} \end{pmatrix} \mid f(t,t^{-1}) = \sum_{i \geq k-m} a_i t^i \right\}.$$

$$\text{As}\begin{pmatrix} t^k & t^{-k}f(t,t^{-1}) \\ 0 & t^{-k}\end{pmatrix}\cdot\begin{pmatrix} g_{11}(t) & g_{12}(t) \\ g_{21}(t) & g_{22}(t)\end{pmatrix} = \begin{pmatrix} t^k g_{11}(t)+t^{-k}g_{21}(t)f(t,t^{-1}) & t^k g_{12}(t)+t^{-k}g_{22}(t)f(t,t^{-1}) \\ t^{-k}g_{21}(t) & t^{-k}g_{22}(t)\end{pmatrix},$$

we see that the elements of MV cycle $\overline{X}_\lambda \cap \mathcal{S}_{\nu_k}$ are given by Laurent polynomials $f(t,t^{-1}) = \sum\limits_{i=k-m}^{2k} a_i t^i$. In particular, $\dim(\overline{X}_\lambda \cap \mathcal{S}_{\nu_k}) = m-k+2k = m+k$, which checks out to be equal to $\rho(\lambda+\nu_k) = 0.5(1\cdot(k+m)+(-1)\cdot(-k-m)) = k+m$. Moreover, it is clear that these varieties are irreducible. So, each weight ν_k with $0 \leq k \leq m$ corresponds to a unique MV cycle.

Awakeness test. *Which irreducible representation of $PGL_2 = SL_2^\vee$ did we just get?*

Answer. $S^m(\mathbb{C}^2) = \mathbb{C}\langle x^m, x^{m-1}y, \ldots, y^m\rangle = \Gamma(\mathbb{P}^1, \mathcal{O}(m)).$

Mirkovic-Vilonen polytopes

Fact. *There exists a very ample line bundle* $\mathcal{L}$ *on* Gr *giving an embedding*

$$\varphi : Gr \hookrightarrow \mathbb{P}(\Gamma(Gr, \mathcal{L})^*)$$

via $\varphi(x) = \{s \in \Gamma(Gr, \mathcal{L}) \mid s(x) = 0\}^*$.

Henceforth we will identify Gr with its image $\varphi(Gr)$ and denote $W := \Gamma(Gr, \mathcal{L})^*$.

Let $T \subset G$ be a maximal torus and $T_K \subset T$ a maximal compact subtorus (for $G = GL_n$, we have $T = (\mathbb{C}^*)^n$ and $T_K = (S^1)^{\times n}$). The torus T_K acts on Gr by conjugation and, hence, on W as well. One can choose an inner product on W invariant under T_K. Better said, an invariant symplectic form on $W_{\mathbb{R}}$ given by

$$\omega(v_1, v_2) := (v_1, iv_2),$$

where $(\cdot, \cdot)$ stands for the chosen inner product.

The action of T_K on W gives rise to a weight decomposition $W = \bigoplus W_\nu$. The moment map for $T_K \curvearrowright Gr$, i.e. $\mu : Gr \to \mathfrak{t}^*_{\mathbb{R}} \simeq \mathfrak{t}_{\mathbb{R}}$ (the last identification is via the Killing form) induced by the action of T_K on Gr is given by

$$\mu(x) = \sum_\nu \frac{|v_\nu|^2}{|v|^2},$$

where (the image under φ of) x is $x = \sum_\nu v_\nu$.

Definition. *The image of a MV cycle under the moment map above is called a* ***Mirkovic-Vilonen polytope****.*

Remark. 1. *The vertices of the MV polytope for MV cycle $\mathcal{S}_\nu \cap \overline{X}_\lambda$ are the points $\mu(t^\eta)$ for coweights $t^\eta \in \mathcal{S}_\nu \cap \overline{X}_\lambda$ (the fixed points for T-action). This easily follows from contractability of $N(\mathcal{K})$ by $2\overset{\vee}{\rho}$-action.*

2. *$\mu(\overline{X}_\lambda) = conv(W \cdot \lambda)$ as the points $t^{W \cdot \lambda}$ are fixed and the images of other fixed points are contained in $conv(W \cdot \lambda)$ (as $X_\eta \subset \overline{X}_\lambda \Leftrightarrow \eta \prec \lambda$).*

The following results are due to Anderson.

Theorem. 1. *If V_λ is an irreducible representation of G. The multiplicity of a ν-weight space is equal to the number of MV polytopes $P_{\nu-\lambda}$ with $P + \lambda \subseteq conv(W \cdot \lambda)$.*

2. *Let V_λ, V_μ be irreducible representations of G and ν a dominant weight. The multiplicity of V_ν in $V_\lambda \otimes V_\mu$ is equal to the number of MV polytopes $P_{\nu-\lambda-\mu}$ with $P + \lambda \subseteq conv(W \cdot \lambda) \cap conv(W \cdot (-\mu) + \nu)$.*

Example. $G = SL_2$ *Let* $\lambda = n, \mu = m$ *with* $m \geq n$ *and* $\nu = k = n + m - 2\ell \geq 0$. *We compute* $conv(W \cdot \lambda) = conv(n, s \cdot n) = conv(n, s(n + 1/2) - 1/2) = conv(n, -n - 1) = [-n - 1, n]$.

$$conv(W \cdot \lambda) = [-n - 1, n].$$

Similarly,

$$conv(W \cdot (-\mu) + \nu) = [k - m, m - 1 + k].$$

The MV polytope P *is the interval* $[k - n - m, 0]$, *which, shifted by* $\lambda = n$, *becomes the interval*

$$P_\nu = [k - m, n].$$

The containment in part (2) *of the Theorem is equivalent to satisfaction of the inequalities* $-n - 1 \leq k - m \leq n \Leftrightarrow m - n - 1 \leq k \leq n + m \Leftrightarrow 0 \leq \ell \leq n$. *This recovers the Clebsch-Gordan rule:*

$$V_m \otimes V_n \simeq \bigoplus_{0 \leq \ell \leq n} V_{m+n-2\ell}.$$