Manipulable outcomes for scoring voting rules

Boris Tsvelikhovskiy

The setup

We suppose that there are m alternatives or candidates for an election.

Definition. *A **profile** is specified by giving an opinion or **preference** (a linear ordering of the alternatives) for each voter.*

Definition. *A **positional** or **scoring** rule is defined for a given m by an m-tuple of numbers $(w_1, w_2, \ldots, w_m)$ (weights) whose entries are in (not necessarily strictly)decreasing order. Alternative k receives score w_i from a voter if k is in position i of his preference order. The total score of candidate k is obtained by summing the scores given to k by each voter. The alternative with highest total score wins.*

Definition. *A profile is said to be **manipulable** if there exists a candidate i such that all members of the electorate for whom he is preferable over the winning alternative can change their preferences in such a way that this candidate wins the election (the preferences of the remaining part of the electorate remain the same).*

We assume that all profiles are equally likely to occur. This probability distribution is known as the Impartial Anonymous Culture (IAC).

Agreement

We will focus on the **asymptotic vulnerability** of positional rules to coalitional manipulation. That is, the limit, as the total number of voters tends to infinity, of the likelihood of observing an anonymous profile at which the corresponding rule is manipulable.

A 'VIQ': Very Important Question

What is the share of manipulable profiles?

$$\frac{\mathcal{MP}}{\mathcal{P}} = ???$$

A plan for obtaining the answer

Step 1. Find (hopefully, a 'digestable') system of linear inequalities, which separate manipulable profiles;

Step 2. Compute the volume of the polytope cut out from the simplex (of all possible profiles) by the inequalities

Remark. *We focus on the first step. Step 2 can be succesfully carried out for 3 candidates and in some cases for 4 (recently). Probabilitistic methods (e.g. Monte-Carlo) produce results with good acuracy.*

An overview of the results

Plurality rule, any number of candidates [Lepelley and Mbih, 1987]

THE PROPORTION OF COALITIONALLY UNSTABLE SITUATIONS UNDER THE PLURALITY RULE

Dominique LEPELLEY

CREMERC, University of Caen, 14032 Caen Cedex, France

Boniface MBIH

University of Yaoundé, Yaoundé, Cameroon
CREMERC, University of Caen, 14032 Caen Cedex, France

Received 6 April 1987

Any positional rule, 3 candidates [Pritchard and Wilson, 2007]

Exact results on manipulability of positional voting rules

Geoffrey Pritchard · Mark C. Wilson

Received: 21 December 2005 / Accepted: 7 December 2006 / Published online: 28 March 2007

Any positional rule, any number of candidates, any 'reasonable' restriction on the coalitions [Diss and T., 2020]

Manipulable outcomes within the class of scoring voting rules

Mostapha Diss   Boris Tsvelikhovskiy

arXiv.org/pdf/1911.09173.pdf

github.com/BorisTsv/VotingManipulability
(Python codes of programs for volume computations)

Some concrete results

Classic

Share of manipulable outcomes with $m = 3$ alternatives

Rule	Manipulable	by Coal_{A_2}	by Coal_{A_3}
Plurality	29.17%	24.65%	15.63%
Antiplurality	51.85%	51.85%	0%
Borda	50.25%	47.71%	9.71%

Interesting Observation

Antiplurality, being the most subjective to manipulability, for the 3-candidate elections, switches to being the least for the 5-candidate elections.

New

Share of manipulable outcomes with $m = 4$ alternatives

Rule	Manipulable	by Coal_{A_2}	by Coal_{A_3}	by Coal_{A_4}
Plurality	87.38%	83.65%	73.87%	63.53%
Antiplurality	87.13%	86.47%	22.83%	0%
Borda	95.65%	95.03%	79.16%	43.38%

The precise value is 87.28%
(El Ouafdi, Lepelley, and Smaoui, 2020)

New

Share of manipulable outcomes with $m = 5$ alternatives

Rule	Manipulable	by Coal_{A_2}	by Coal_{A_3}	by Coal_{A_4}	by Coal_{A_5}
Plurality	99.51%	99.37%	99.05%	98.57%	98.04%
Antiplurality	97.15%	96.79%	54.78%	6.52%	0%
Borda	99.76%	99.23%	98.95%	98.18%	97.83%

Other possible applications

- Computations of other indices of manipulability.
- Carry out the computations for restricted coalitions (we did for 'same first candidate in preferences').
- Our approach for obtaining inequalities may work for other rules (e. g. iterative scoring rules, also known as multi-stage or sequential elimination scoring rules. This class of multi-round procedures are based on the same scoring principle but proceed by eliminating one or more alternatives at each round, until there is only one alternative left who is considered as the winner).

What do these inequalities look like?

$$\begin{cases} \text{Preliminary Inequalities (PI)} \\ \sum_{i=1}^{m!} p_i = 1 \\ p_i \geq 0 \\ \text{the initial arrangement is } (A_1, A_2, \ldots, A_m) \end{cases}$$

$$\begin{cases} \text{Strategic Inequalities (SI)} \\ \sum_{i \neq k} d(A_k, A_i) > (\lambda_2 + \ldots + \lambda_{m-1} + \lambda_m)\mathfrak{C}_{\mathcal{A}_k} \\ \sum_{i \neq k} d(A_k, A_i) - M_1 > (\lambda_3 + \ldots + \lambda_{m-1} + \lambda_m)\mathfrak{C}_{\mathcal{A}_k} \\ \ldots \\ \sum_{i \neq k} d(A_k, A_i) - \sum_{j=1}^{m-4} M_j > (\lambda_{m-2} + \lambda_{m-1} + \lambda_m)\mathfrak{C}_{\mathcal{A}_k} \\ \sum_{i \neq k} d(A_k, A_i) - \sum_{j=1}^{m-3} M_j > (\lambda_{m-1} + \lambda_m)\mathfrak{C}_{\mathcal{A}_k} \\ M_{m-1} > \lambda_m \mathfrak{C}_{\mathcal{A}_k}, \end{cases}$$