Fall 2025

MATH 007A: Calculus for Life Sciences I

Midterm 2 Review

Solutions

Problem 1. Find the derivatives of the following functions. **Do not simplify your answers.**

(a) $f(x) = \frac{x^2 + \frac{1}{x^2}}{1-5x}$

Solution. Use the quotient rule with $u = x^2 + \frac{1}{x^2}$ and $v = 1 - 5x$:

$$u' = 3x^2 - 2x^3 \quad v' = -5$$
$$f'(x) = \frac{\left(2x - \frac{2}{x^3}\right)(1-5x) - \left(x^2 + \frac{1}{x^2}\right)(-5)}{(1-5x)^2}.$$

(b) $g(x) = \sqrt{4x^2 + 5x - 7}$

Solution. Use the chain rule:

$$g'(x) = 0.5(4x^2 + 5x - 7)^{-0.5} \cdot (4x^2 + 5x - 7)' = \frac{8x + 5}{2\sqrt{4x^2 + 5x - 7}}.$$

Problem 2. Find the derivatives of the following functions. **Do not simplify your answers.**

(a) $f(x) = 2^x \cdot e^{5x}$

Solution. Use product rule with $u = 2^x$ and $v = e^{5x}$:

$$u' = \ell n(2)2^x \quad v' = 5e^{5x}$$
$$f'(x) = \ell n(2)2^x \cdot e^{5x} + 2^x \cdot 5e^{5x}.$$

(b) $h(x) = \ell n\left(\frac{x-5}{x^2+8}\right)$

Solution. Use chain and quotient rules:

$$h'(x) = \frac{1}{\frac{x-5}{x^2+8}} \cdot \left(\frac{x-5}{x^2+8}\right)' = \frac{x^2+8}{x-5} \cdot \frac{1 \cdot (x^2+8) - (x-5) \cdot 2x}{(x^2+8)^2} = \frac{(x^2+8) - (x-5) \cdot 2x}{(x-5)(x^2+8)}.$$

Alternatively, we can use logarithmic properties to write:

$$h(x) = \ell n\left(\frac{x-5}{x^2+8}\right) = \ell n(x-5) - \ell n(x^2+8),$$

and then find the derivative:

$$h'(x) = \frac{1}{x-5} - \frac{2x}{x^2+8} = \frac{(x^2+8) - (x-5) \cdot 2x}{(x-5)(x^2+8)}.$$

(c) $m(x) = e^{\ell n(x^3 - x)}$

Solution. Since the exponential function e^x and the natural logarithm $\ell n(x)$ are inverse functions, we can simplify the composition:

$$m(x) = e^{\ell n(x^3-x)} = x^3 - x$$

Now we differentiate using the power rule:

$$m'(x) = (x^3 - x)' = 3x^2 - 1.$$

Problem 3. Consider the function $f(x) = \sec(x) = \frac{1}{\cos(x)}$.

(a) Show that $f'(x) = \sec(x)\tan(x)$.

Solution. Use chain rule to obtain:

$$f'(x) = \left(\frac{1}{\cos(x)}\right)' = -\frac{1}{\cos^2(x)} \cdot (\cos(x))' = -\frac{-\sin(x)}{\cos^2(x)} = \frac{\sin(x)}{\cos^2(x)} = \frac{1}{\cos(x)} \cdot \frac{\sin(x)}{\cos(x)} = \sec(x)\tan(x).$$

(b) Find $f''(x)$.

Solution. Use the identity in (a), product and chain rules:

$$f''(x) = (\sec(x)\tan(x))' = (\sec(x))'\tan(x) + \sec(x)(\tan(x))' = \sec(x)\tan^2(x) + \sec^3(x).$$

Problem 4. Consider the curve given by the equation $y\cos(x) = x^2 + y^2$.

(a) Use implicit differentiation to compute $\frac{dy}{dx}$.

Solution. Differentiate the left-hand side using the product rule:

$$\frac{d}{dx}(y\cos(x)) = \frac{dy}{dx} \cdot \cos(x) + y \cdot \frac{d}{dx}(\cos(x)) = \cos(x)\frac{dy}{dx} - y\sin(x).$$

Differentiate the right-hand side:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 2x + 2y\frac{dy}{dx},$$

where we used the chain rule for differentiating y^2, treating y as a function of x.

Putting both sides together we obtain the equation

$$\cos(x)\frac{dy}{dx} - y\sin(x) = 2x + 2y\frac{dy}{dx}.$$

Now we solve for $\frac{dy}{dx}$. First, bring all terms involving $\frac{dy}{dx}$ to one side:

$$\cos(x)\frac{dy}{dx} - 2y\frac{dy}{dx} = 2x + y\sin(x).$$

Factor out $\frac{dy}{dx}$ on the left-hand side:

$$(\cos(x) - 2y)\frac{dy}{dx} = 2x + y\sin(x).$$

Finally, divide both sides by $\cos(x) - 2y$ to isolate $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{2x + y\sin(x)}{\cos(x) - 2y}.$$

(b) Verify that the point $(0, 1)$ lies on the curve.

Solution. Plug the coordinates into the original equation:

$$y\cos(x) = x^2 + y^2 \Leftrightarrow 1 \cdot \cos(0) = 0^2 + 1^2 \Leftrightarrow 1 = 1\checkmark$$

(c) Determine the equation of the tangent line to the curve at the point $(0, 1)$.

Solution. Use the expression for $\frac{dy}{dx}$ obtained in (a):

$$\frac{2x + y\sin(x)}{\cos(x) - 2y} = \frac{0 + \sin(0)}{\cos(0) - 2} = \frac{0}{1 - 2} = 0.$$

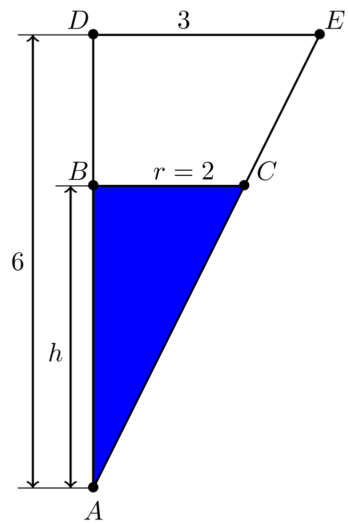
So the tangent line is horizontal at $(0, 1)$, the equation is $y = 1$.

Problem 5. The local aquarium has a cool new fish tank. This tank is not a rectangular prism like most fish tanks are. Instead, it's an inverted right circular cone i.e. an upside down cone. Water is being pumped into the tank a rate of 5 cubic feet per minute. The height of the tank is 6 feet, and the radius of the circular opening at the top is 3 feet. What is the rate at which the water level in the tank is rising when the depth of the water in the tank is 2 feet?

Note: the volume of a right circular cone with radius r and height h is equal to $\dfrac{1}{3}\pi r^2 h$.

Solution. We first notice that both $h(t)$ and $r(t)$ are functions of time t, and we are asked to find the rate at which the water level is rising, i.e., $\frac{dh(t)}{dt}$, when the depth is 2 feet.

Because the tank is a cone and water forms a smaller, similar cone inside it, the triangles formed by the radius and height at any time t are similar ($\triangle ABC \sim \triangle ADE$):



So we have the proportion:

$$\frac{h(t)}{r(t)} = \frac{6}{3} = 2 \quad \Rightarrow \quad r(t) = \frac{h(t)}{2}.$$

Using this relationship, we can express the volume of water in the tank as a function of height $h(t)$ alone:

$$V(t) = \frac{1}{3}\pi r^2(t)h(t) = \frac{1}{3}\pi\left(\frac{h(t)}{2}\right)^2 h(t) = \frac{1}{12}\pi h^3(t)$$

Now we differentiate both sides with respect to time t:

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{1}{12}\pi h^3(t)\right) = \frac{3}{12}\pi h^2(t)\cdot\frac{dh}{dt} = \frac{\pi}{4}h^2(t)\cdot\frac{dh}{dt}$$

We are given that water is being pumped in at a rate of $\frac{dV}{dt} = 5$ ft^3/min and the depth at that moment is $h(t) = 2$ ft. Plugging these values in the above equation gives

$$5 = \frac{\pi}{4}\cdot 2^2\cdot\frac{dh}{dt} \quad \Rightarrow \quad 5 = \pi\cdot\frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = \frac{5}{\pi}\text{ ft/min}$$

Problem 6. The volume of a cube is increasing at a rate of $5\,\text{in}^3/\text{sec}$. How fast is the surface area increasing when the length of an edge is $10\,\text{in}$?

Hint. The surface area of a cube with edge length x is $6x^2$.

Solution. Let $x(t)$ be the length of a cube's edge at time t. Then the volume and surface area are $V(t) = x^3(t)$ and $S(t) = 6x^2(t)$. We are given that $\frac{dV(t)}{dt} = V'(t) = 5$ in^3/sec. To find $\frac{dS(t)}{dt}$, we use the chain rule:

$$\frac{dS(t)}{dt} = (6x^2(t))' = 12x(t)\cdot\frac{dx(t)}{dt}$$

We now find $\frac{dx(t)}{dt}$ using the equation for the derivative of the volume:

$$\frac{dV(t)}{dt} = (x^3(t))' = 3x^2(t) \cdot \frac{dx(t)}{dt}$$

Solving for $\frac{dx}{dt}$ produces:

$$\frac{dx}{dt} = \frac{1}{3x^2(t)} \cdot \frac{dV(t)}{dt}.$$

Plugging in the given values of $x(t) = 10$ and $\frac{dV}{dt} = 5$ allows to find $\frac{dx(t)}{dt} = \frac{5}{3 \cdot 10^2} = \frac{1}{60}$. Finally,

$$\frac{dS(t)}{dt} = 12x(t) \cdot \frac{dx(t)}{dt} = 12 \cdot 10 \cdot \frac{1}{60} = 2 \text{ in}^2/\text{sec}.$$