

Final Review**Solutions**

Problem 1. Find the global maximum and minimum of the function $f(x) = 27x - x^3$ on the closed interval $[-4, 0]$.

Solution. First, we find the critical points by computing the derivative and solving $f'(x) = 0$:

$$f'(x) = 27 - 3x^2,$$

$$0 = 27 - 3x^2,$$

$$3x^2 = 27,$$

$$x^2 = 9,$$

$$x = \pm 3.$$

The critical points are $x = 3$ and $x = -3$. Only $x = -3$ lies within the interval $[-4, 0]$, so we consider it along with the endpoints $x = -4$ and $x = 0$.

Now we evaluate the function at these points:

$$f(-4) = 27 \cdot (-4) - (-4)^3 = -108 + 64 = -44,$$

$$f(-3) = 27 \cdot (-3) - (-3)^3 = -81 + 27 = -54,$$

$$f(0) = 27 \cdot 0 - 0^3 = 0.$$

Conclusion: Among the values $\{-44, -54, 0\}$, the maximum is 0 and the minimum is -54 . Therefore, the function $f(x)$ attains a **global maximum** of 0 at $x = 0$, and a **global minimum** of -54 at $x = -3$ on the interval $[-4, 0]$.

Problem 2. We continue working with the function $f(x) = 27x - x^3$.

(a) Compute the average rate of change in $f(x)$ between $x = -4$ and $x = 0$.

Solution. The average rate of change is given by

$$\frac{f(0) - f(-4)}{0 - (-4)} = \frac{0 - (-44)}{4} = \frac{44}{4} = 11.$$

(b) Find all numbers $-4 < c < 0$ with $f'(c)$ equal to the number you found in (a).

Solution. We previously found that

$$f'(x) = 27 - 3x^2.$$

Set this equal to the average rate of change:

$$27 - 3c^2 = 11,$$

$$3c^2 = 16,$$

$$c^2 = \frac{16}{3},$$

$$c = \pm \sqrt{\frac{16}{3}} = \pm \frac{4}{\sqrt{3}}.$$

Only the negative value lies in the interval $(-4, 0)$, so $c = -\frac{4}{\sqrt{3}}$.

Problem 3. Consider the function $f(x) = 2 - 12x + x^3$.

(a) Find the intervals on which $f(x)$ is increasing or decreasing.

Solution. We compute the first derivative:

$$f'(x) = -12 + 3x^2 = 3(x^2 - 4) = 3(x - 2)(x + 2).$$

Critical points: $x = 2$ and $x = -2$.

Sign chart:

- $x < -2$: $f'(x) > 0 \rightarrow$ increasing
- $-2 < x < 2$: $f'(x) < 0 \rightarrow$ decreasing
- $x > 2$: $f'(x) > 0 \rightarrow$ increasing

Answer: increasing on $(-\infty, -2) \cup (2, \infty)$; decreasing on $(-2, 2)$.

(b) Find the critical point(s) of $f(x)$ and classify each as a local maximum, local minimum, or neither. Write each point of local max/min as an ordered pair (x, y) .

Solution. First we find the x -coordinates of critical points from the equation

$$f'(x) = -12 + 3x^2 = 3(x^2 - 4) = 3(x - 2)(x + 2) = 0.$$

Critical points: $x = 2$ and $x = -2$.

- At $x = -2$, $f(x)$ changes from increasing to decreasing \rightarrow local max.
- At $x = 2$, $f(x)$ changes from decreasing to increasing \rightarrow local min.

$$f(-2) = 2 + 24 - 8 = 18, \quad f(2) = 2 - 24 + 8 = -14.$$

Answer: $f(x)$ has a local max at $(-2, 18)$; a local min at $(2, -14)$.

(c) Find the intervals on which $f(x)$ is concave up or concave down.

Solution. We compute the second derivative:

$$f''(x) = (-12 + 3x^2) = 6x.$$

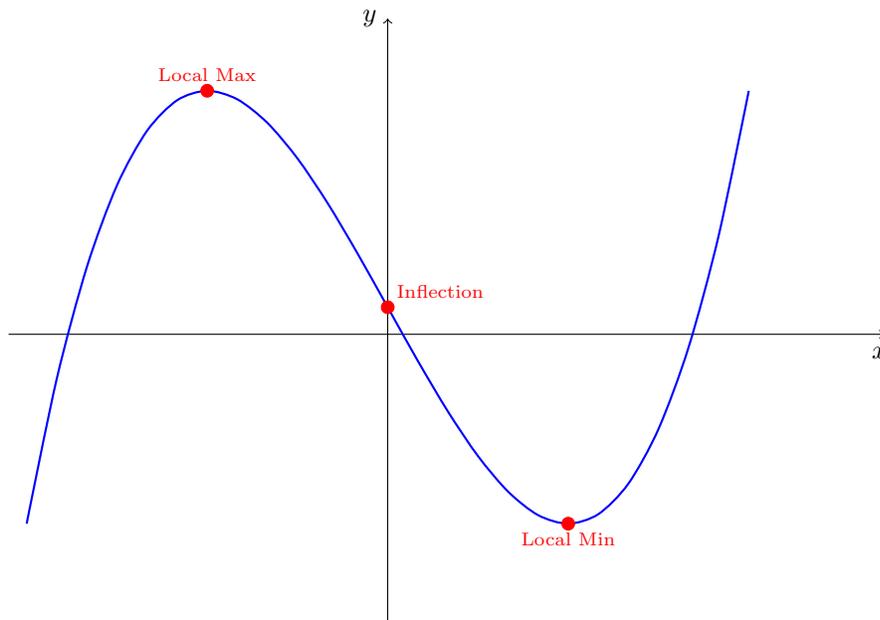
- $f''(x) < 0$ if $x < 0$: concave down
- $f''(x) > 0$ if $x > 0$: concave up

Answer: $f(x)$ is concave down on $(-\infty, 0)$; concave up on $(0, \infty)$.

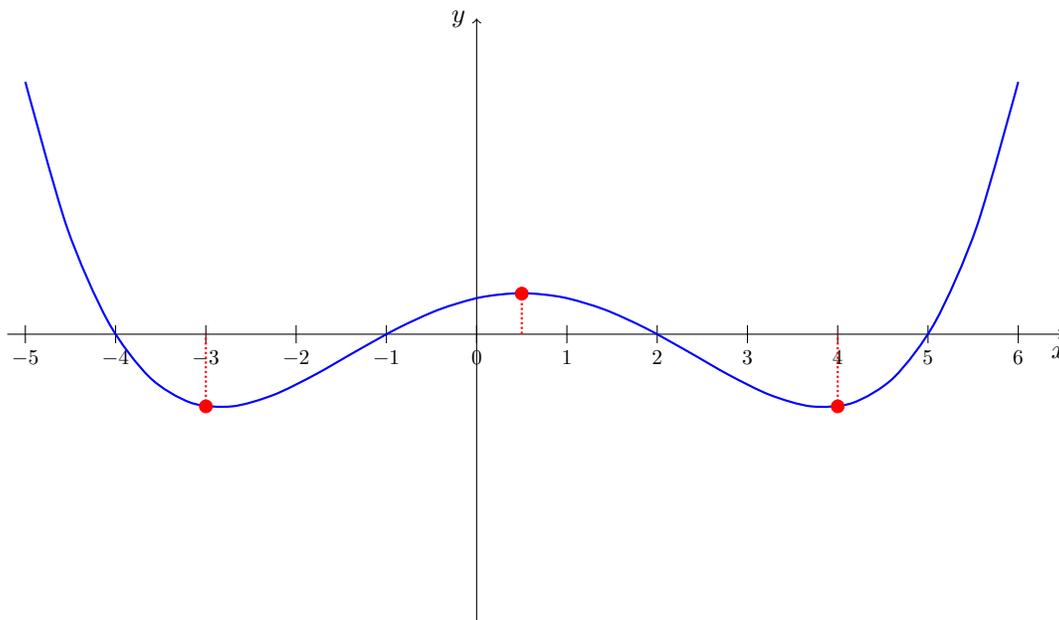
(d) Find all points of inflection of f . Write each point as an ordered pair (x, y) .

Solution. We solve the equation $f''(x) = 0$ to find that the only potential inflection point is at $x = 0$. Since $f''(x)$ changes sign at $x = 0$, the point $(0, f(0)) = (0, 2)$ is, indeed, an inflection point.

(e) Sketch the graph of $f(x)$ on the axes provided below. Label extrema and inflection point(s).



Problem 4. The graph below shows the **derivative** $f'(x)$ of a function $f(x)$. Use the graph to estimate the x -coordinates of all inflection points of $f(x)$.



Solution. As we discussed, inflection points of a function $f(x)$ occur where the concavity changes, which happens when the second derivative $f''(x)$ changes sign. Since $f''(x)$ is the derivative of $f'(x)$, this corresponds to local extrema (maxima or minima) of the graph of $f'(x)$.

From the graph of $f'(x)$, we observe local extrema at approximately $x = -3$, $x = 0.5$, and $x = 4$. Therefore, the function $f(x)$ has inflection points approximately at $x = -3$, $x = 0.5$, and $x = 4$.

Problem 5. Use L'Hôpital's Rule to evaluate the following limits:

(a) $\lim_{x \rightarrow 0} \frac{e^{5x} - 1}{x}$

Solution. This is an indeterminate form of type $\frac{0}{0}$, since:

$$\lim_{x \rightarrow 0} e^{5x} = 1 \quad \text{and} \quad \lim_{x \rightarrow 0} x = 0.$$

So we apply L'Hôpital's Rule:

$$\lim_{x \rightarrow 0} \frac{e^{5x} - 1}{x} = \lim_{x \rightarrow 0} \frac{(e^{5x} - 1)'}{x'} = \lim_{x \rightarrow 0} \frac{5e^{5x}}{1} = 5e^0 = 5.$$

(b) $\lim_{x \rightarrow \infty} \frac{\ln(x)}{x}$

Solution. This is an indeterminate form of type $\frac{\infty}{\infty}$, since:

$$\lim_{x \rightarrow \infty} \ln(x) = \infty, \quad \lim_{x \rightarrow \infty} x = \infty.$$

So we apply L'Hôpital's Rule:

$$\lim_{x \rightarrow \infty} \frac{\ln(x)}{x} = \lim_{x \rightarrow \infty} \frac{(\ln(x))'}{(x)'} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0.$$

(c) $\lim_{x \rightarrow 0} \frac{x^2}{1 - \cos(x)}$

Solution. This is an indeterminate form of type $\frac{0}{0}$, since:

$$\lim_{x \rightarrow 0} x^2 = 0, \quad \lim_{x \rightarrow 0} (1 - \cos(x)) = 1 - 1 = 0.$$

Apply L'Hôpital's Rule:

$$\lim_{x \rightarrow 0} \frac{x^2}{1 - \cos(x)} = \lim_{x \rightarrow 0} \frac{2x}{\sin(x)}.$$

Now observe that

$$\lim_{x \rightarrow 0} \frac{2x}{\sin(x)} = 2 \cdot \lim_{x \rightarrow 0} \frac{x}{\sin(x)} = 2 \cdot 1 = 2.$$

Remark. Alternatively, instead of using the identity $\lim_{x \rightarrow 0} \frac{x}{\sin(x)} = 1$ directly, we could apply L'Hôpital's Rule again:

$$\lim_{x \rightarrow 0} \frac{2x}{\sin(x)} = \lim_{x \rightarrow 0} \frac{(2x)'}{(\sin(x))'} = \lim_{x \rightarrow 0} \frac{2}{\cos(x)} = \frac{2}{\cos(0)} = \frac{2}{1} = 2.$$