Summer 2025

MATH 149A: PROBABILITY AND MATHEMATICAL STATISTICS

Midterm Review Problems
Solutions

Probability distributions on finite sets

1. You have a bag with 4 red marbles and 3 blue marbles. What is the probability of drawing a red marble?

   Solution: the probability of drawing a red marble is $\frac{4}{7}$.

2. If you flip a fair coin 5 times, what is the probability of getting exactly 2 heads?

   Solution: the probability of getting exactly 2 heads in 5 coin flips is $\frac{\binom{5}{2}}{2^5} = \frac{10}{32} = \frac{5}{16}$.

3. If you flip a fair coin 10 times, what is the probability of getting at least 2 heads?

   Solution: the probabilities of getting 0 or 1 heads are $\frac{1}{2^{10}}$ and $\frac{10}{2^{10}}$. Therefore, the probability of getting at least 2 heads is $1 - \frac{1}{2^{10}} - \frac{10}{2^{10}} = \frac{1013}{1024}$.

4. Among 35 students in a class, 17 earned 'A' on the midterm, 14 earned 'A' on the final exam, and 11 did not earn 'A' on either exam. What is the probability that a randomly selected student from this class earned 'A' on both exams?

   Solution: let us define the following events:

$$\begin{aligned} A &= \text{student earned an 'A' on the midterm,} \\ B &= \text{student earned an 'A' on the final,} \\ C &= \text{student earned no 'A' on either exam.} \end{aligned}$$

   We are asked to compute $P(A \cap B)$, the probability that a student earned an 'A' on *both* exams.

   From the problem statement, we know:

$$P(A) = \frac{17}{35}, \quad P(B) = \frac{14}{35}, \quad P(C) = \frac{11}{35}.$$

   Since students who earned an 'A' on at least one exam are the complement of those in C, we have:

$$P(A \cup B) = 1 - P(C) = 1 - \frac{11}{35} = \frac{24}{35}.$$

   Now we use the inclusion-exclusion formula to get:

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = \frac{17}{35} + \frac{14}{35} - \frac{24}{35} = \frac{7}{35}.$$

Counting principles

1. How many different words can be produced from the letters of the word '`MISSISSIPPI`'?

   Solution: the word `MISSISSIPPI` has 11 letters in total with the following frequencies:

   - The letter `I` appears 4 times,
   - The letter `S` appears 4 times,
   - The letter `P` appears 2 times,

- The letter `M` appears 1 time.

The number of distinct permutations of these letters, is equal to $\frac{11!}{1! \cdot 4! \cdot 4! \cdot 2!}$.

2. In a department consisting of 10 students and 8 faculty members, a committee is to be formed. The committee will consist of a president and a vice president, both of whom must be chosen from the faculty members. Additionally, 3 students will be chosen as members of the committee. How many different committees can be formed?

   Solution: to form the committee, we have two distinct tasks.

   Task 1: Choose a president and a vice president from the 8 faculty members. This can be done in $P(8, 2) = 8 \cdot 7 = 56$ ways.

   Task 2: Select 3 students from the pool of 10 available. This can be done in $\binom{10}{3}$ ways.

   By the multiplication principle, the total number of ways to form the committee is $P(8, 2) \cdot \binom{10}{3} = 56 \cdot 120 = 6720$.

   Therefore, there are 6720 different committees that can be formed.

3. A password consists of 4 letters (A-Z) followed by 3 digits (0-9). How many different passwords are possible?

   Solution: each password is made up of two parts:

   - The first part consists of 4 letters, where each letter can be any of the 26 letters in the English alphabet (A–Z).
   - The second part consists of 3 digits, where each digit can be any of the 10 digits from 0 to 9.

   Since each character is chosen independently and repetition is allowed, the total number of possible passwords is $26^4 \cdot 10^3$.

Conditional probability and Bayes' formula

1. A pair of fair six-sided dice is rolled once.

   (a) Find the probability that there is at least one six.
   Solution: the probability of getting at least one six is $1 - \left(\frac{5}{6}\right)^2 = \frac{11}{36}$.

   (b) Find the probability that both dice show sixes, given that there is at least one six.
   Solution: given that there is at least one six, the probability of getting two sixes is $\frac{1}{6}$.

2. A factory produces two types of widgets: Type A and Type B. Type A accounts for 10% of total production, while the remaining 90% are Type B. Of the Type A widgets, 95% pass quality control, while 90% of Type B widgets pass.

   If a randomly selected widget has passed quality control, what is the probability that it is of Type A?

   Solution: let A be the event that a randomly selected widget is of Type A, and let P be the event that a widget passes quality control. Since all widgets are either Type A or Type B, the event that a widget is of Type B is A^c, the complement of A.

   We are asked to compute $P(A \mid P)$, the probability that a widget is of Type A given that it passed quality control.

   Using Bayes' Theorem:

$$P(A \mid P) = \frac{P(P \mid A) \cdot P(A)}{P(P \mid A) \cdot P(A) + P(P \mid A^c) \cdot P(A^c)} = \frac{0.95 \cdot 0.1}{0.95 \cdot 0.1 + 0.9 \cdot 0.9}.$$

3. An airport security system correctly detects 99% of explosive materials, but it also gives false alarms for 2% of non-explosive items. Out of all checked items, 5% contain explosives. If a randomly selected item sets off the alarm, what is the probability that it actually contains explosives?

   Solution: Let E be the event of the item containing explosives and A be the event of the alarm going off. We want to find $P(E|A)$. Using Bayes' formula, we get

$$P(E|A) = \frac{P(A|E)P(E)}{P(A|E)P(E) + P(A|E^c)P(E^c)} = \frac{0.99 \cdot 0.05}{0.99 \cdot 0.05 + 0.02 \cdot 0.95}.$$

Independence

1. Consider two events A and B such that $P(A) = 0.3$ and $P(B) = 0.4$. If A and B are independent, what is $P(A \cap B)$?

   Solution: if A and B are independent, then $P(A \cap B) = P(A) \cdot P(B) = 0.3 \cdot 0.4 = 0.12$.

2. Consider two events E and F such that $P(E) = 0.6$ and $P(F) = 0.7$. If E and F are independent, what is $P(E \cup F)$?

   Solution: if E and F are independent, then $P(E \cup F) = P(E) + P(F) - P(E \cap F) = 0.6 + 0.7 - P(E) \cdot P(F) = 0.6 + 0.7 - 0.6 \cdot 0.7 = 1.3 - 0.42 = 0.88$.

3. In a survey, it was revealed that 75% of people like spicy food, and 20% of people are vegetarians. Among those who like spicy food, 10% are vegetarians. Are the events "liking spicy food" and "being a vegetarian" independent?

   Solution: let S be the event that a person likes spicy food and V be the event that a person is a vegetarian.

   We are given:

$$P(S) = 0.75, \quad P(V) = 0.20, \quad P(V \mid S) = 0.10.$$

   To check for independence, we compare $P(V \mid S)$ with $P(V)$:

$$P(V \mid S) = 0.10 \neq 0.20 = P(V).$$

   Therefore, the events "liking spicy food" and "being a vegetarian" are **not independent**.

4. You have a deck of 52 cards. Events J and K are defined as follows:

$$J = \text{Drawing a red card}$$
$$K = \text{Drawing a face card}$$

   Are these events independent?

   Solution: as $P(J) = \frac{1}{2}, P(K) = \frac{12}{52}$ with $P(J) \cdot P(K) = \frac{6}{52} = P(J \cap K)$, the events are independent.

Random variables, PMF, PDF, CDF and expectation

1. Let X be a random variable representing the number of heads obtained when flipping a coin 3 times. Find the probability mass function of X.

   Solution: the probability mass function of X is given by:

$$P(X = k) = \binom{3}{k} \left(\frac{1}{2}\right)^3 = \frac{\binom{3}{k}}{8} \quad \text{for } k = 0, 1, 2, 3.$$

k	$P(X=k)$
0	$\frac{1}{8}$
1	$\frac{3}{8}$
2	$\frac{3}{8}$
3	$\frac{1}{8}$

2. Let Y be the random variable representing the number of times a 6 appears when a fair six-sided die is rolled 4 times.

   (a) Find the PMF of Y.
   Solution: this is a binomial distribution with probability mass function given by:

$$P(Y=k)=\binom{4}{k}\cdot\frac{1^k\cdot 5^{4-k}}{6^4}\quad \text{for } k=0,1,2,3,4.$$

k	$P(Y=k)$
0	$\frac{625}{1296}$
1	$\frac{500}{1296}$
2	$\frac{150}{1296}$
3	$\frac{20}{1296}$
4	$\frac{1}{1296}$

   (b) Find the expected value of Y.
   Solution: The expected value of Y is given by:

$$\mathbb{E}(Y)=\sum_{k=0}^{4}k\cdot P(Y=k)=0\cdot\frac{625}{1296}+1\cdot\frac{500}{1296}+2\cdot\frac{150}{1296}+3\cdot\frac{20}{1296}+4\cdot\frac{1}{1296}=\frac{864}{1296}=\frac{2}{3}.$$

3. Suppose that the sample space S contains four elements $\{-1,1,2,3\}$, with probabilities 0.1, 0.4, 0.2, and 0.3 respectively. Suppose $X(s)=s^2-4$ for $s\in S$. Compute the expected value $\mathbb{E}(X)$.

   Solution: we compute $X(-1)=(-1)^2-4, X(1)=1^2-4=-3, X(2)=2^2-4=0$ and $X(3)=3^2-4=5$. Therefore the PMF of X is $P(X=-3)=0.1+0.4=0.5, P(X=0)=0.2$ and $P(X=5)=0.3$. The expected value of X is

$$\mathbb{E}(X)=0.5\cdot(-3)+0.2\cdot 0+0.3\cdot 5=0.$$

4. Let X be a random variable whose PDF is given by

$$f(x)=\begin{cases} c & \text{if } -3\le x\le 1,\\ 0 & \text{else.}\end{cases}$$

   (a) Determine the value of c.

   Solution: we must have $1=\int\limits_{-\infty}^{\infty} f(x)\,dx=\int\limits_{-3}^{1} c\,dx=4c$, so $c=\frac{1}{4}$.

   (b) Compute the expected value $\mathbb{E}(X)$.

   Solution: since $c=\frac{1}{4}$, we have $\mathbb{E}(X)=\int\limits_{-\infty}^{\infty} xf(x)\,dx=\int\limits_{-3}^{1}\frac{x}{4}\,dx=\frac{x^2}{8}\Big|_{-3}^{1}=\frac{1}{8}-\frac{9}{8}=-1.$

   (c) Let $Y=5-\frac{X}{2}$ and find the CDF of Y.
   Solution: we start by expressing the CDF of Y in terms of the CDF of X:

$$F_Y(y)=P(Y\le y)=P\left(5-\frac{X}{2}\le y\right)=P\left(-\frac{X}{2}\le y-5\right)=P\left(X\ge 10-2y\right).$$

Then, $F_Y(y) = 1 - P(X < 10 - 2y) = 1 - F_X(10 - 2y) =$

$$\begin{cases} 0, & \text{if } 10 - 2y > 1 \Leftrightarrow y < 4.5, \\ 1 - \frac{10-2y+3}{4} = \frac{y}{2} - \frac{9}{4}, & \text{if } -3 \leq 10 - 2y \leq 1 \Leftrightarrow 4.5 \leq y \leq 6.5, \\ 1, & \text{if } 10 - 2y < -3 \Leftrightarrow y > 6.5. \end{cases}$$