MATH 149A: Probability and Mathematical Statistics

Homework 4

Marginal distributions

Problem 1. Imagine a library with sixteen books, each labeled with a genre and an author. Let X represent the genre, and Y represent the author of the selected book. The joint probability mass function is given by the table below.

x/y	Fiction	Detective Stories	Science	Poetry
Author A	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
Author B	$\frac{3}{16}$	0	0	$\frac{1}{8}$
Author C	0	$\frac{1}{16}$	$\frac{3}{16}$	0
Author D	0	0	0	$\frac{1}{4}$

(a) (5 points) What is the probability that a randomly chosen book is a Detective story by author C?

(b) (2 points) Name a well-known British author who gained fame for writing detective stories and whose last name starts with 'C.'

(c) (4 points) Compute the marginal PMF of X.

(d) (4 points) Compute the marginal PMF of Y.

Transformations of pairs of random variables

Problem 2. Consider two random variables X_1 and X_2 with joint PDF

$$f(x_1, x_2) = \begin{cases} 2x_1^2 x_2 + 2x_2^2, & 0 \le x_1 \le 1, 0 \le x_2 \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

and a transformation $u : (X_1, X_2) \to (Y_1, Y_2)$ given by $u(X_1, X_2) = (X_1 X_2, X_2^3)$.

(a) (5 points) Find the inverse transformation to u, i.e. $w : (Y_1, Y_2) \to (X_1, X_2)$ with $w \circ u = \mathrm{id}_{(X_1, X_2)}$.

(b) (10 points) Compute the Jacobian of w.

(c) (5 points) Determine the joint PDF of Y_1 and Y_2.

Conditional distribution and expectation

Problem 3. Consider two random variables X and Y with joint PDF

$$f(x, y) = \begin{cases} 6y & 0 < y < x < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) (5 points) Find the conditional PDF of Y, given $X = x$.

(b) (5 points) Use your answer in (a) to find the conditional PDF of Y, given $X = 0.1$.

(c) (5 points) Find the conditional mean of Y, given $X = x$.

(d) (5 points) Use your answer in (c) to find the conditional mean of Y, given $X = 0.5$.

Independent random variables, covariance and correlation

Problem 4. Let X and Y have the joint PMF

(x, y)	$(1,1)$	$(1,2)$	$(1,3)$	$(2,1)$	$(2,2)$	$(2,3)$
$p(x, y)$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{4}{12}$	$\frac{3}{12}$	$\frac{1}{12}$	$\frac{1}{12}$

and $p(x, y)$ is equal to zero elsewhere.

(a) (5 points) Find the means $\mathbb{E}(X)$, $\mathbb{E}(Y)$ and $\mathbb{E}(XY)$.

(b) (5 points) Find the covariance of X and Y.

(c) (5 points) Find the variances of X and Y.

(d) (5 points) Find the correlation coefficient $\rho(X, Y)$.

Problem 5. Let X and Y be two random variables whose PDF is given by

$$f(x, y) = \begin{cases} 8xy, & \text{if } 0 \leq y \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (5 points) Find the conditional density function of Y conditioned on X.

(b) (5 points) Use your answer in (a) to find the conditional PDFs of Y given $X = 0.3$ and $X = 0.7$.

(c) (5 points) Are X and Y independent? Provide justification for your answer, as a correct response without explanation will not receive points.

Problem 6. (10 points) The random variables X and Y with range $\{1, 2, 3\}$ have uniform joint PMF $p(x, y) = \frac{1}{9}$. Check if X and Y are independent.

Problem 6. The goal of this problem is is to illustrate that although the covariance between any two independent random variables is zero:

$$\text{X and Y are independent} \implies \mathrm{Cov}(X, Y) = 0,$$

the reverse statement is not necessarily true.

Let the random variables X and Y with range $\{-1, 0, 1\}$ have the joint PMF

$$p(x, y) = \begin{cases} 0.25, & (x, y) \in \{(0, 1), (0, -1), (1, 0), (-1, 0)\} \\ 0, & \text{elsewhere.} \end{cases}$$

(a) (5 points) Compute the expected values $\mathbb{E}(X), \mathbb{E}(Y), \mathbb{E}(XY)$.

(b) (5 points) Find the covariance between X and Y.

(c) (5 points) Explain why X and Y are not independent.