Summer 2025

MATH 149A: Probability and Mathematical Statistics

Homework 3

Problem 1. Let T be a continuous random variable with probability density function

$$f_T(t) = \begin{cases} c(4-t^2), & \text{for } 0 \le t \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (3 points) Determine the value of c that makes $f_T(t)$ a valid probability density function.

(b) (3 points) Compute the expected value $\mathbb{E}(T)$.

(c) (3 points) Compute the variance $\mathrm{Var}(T)$.

(d) (1 point) Find the standard deviation of T.

Problem 2. Consider a class of students who have taken a MATH 149A exam. The exam scores are distributed between 0 and 100 with the average score of 70 points.

(a) (2 points) Use Markov's inequality to provide an upper bound on the probability that a randomly chosen student scored at least 90 points.

(b) (3 points) Now suppose the variance of the exam scores is known to be 25. Use Chebyshev's inequality to provide a (potentially tighter) upper bound for the probability that a student scored 90 or more.

Hint. You may use the version of Chebyshev's inequality stated in the remark on page 3 of the Lecture 11 notes.

Problem 3. (5 points) Consider a movie being independently rated by two reviewers on a scale from 1 to 5. Let X represent the rating given by the first reviewer, assumed to be uniformly distributed between 1 and 5. The second reviewer, after observing the first rating, decides to assign a rating equal to or higher than the first one, with this rating uniformly distributed between the value given by the first reviewer and 5. Let Y represent the rating given by the second reviewer, assumed to be uniformly distributed between X and 5. Provide a table for the values of the joint probability mass function of X and Y.

Problem 4. Consider random variables (X, Y) with a joint probability density function given by

$$f(x,y) = \begin{cases} 4x - 2y & \text{if } 0 \le x \le 1, 0 \le y \le x, \\ 0 & \text{otherwise.} \end{cases}$$

(a) (5 points) Check that $f(x, y)$ is a valid probability density function.

(b) (5 points) Set up the double integral to compute $P(X + Y > 1)$. Draw the unit square and clearly indicate the region of integration.

(c) (5 points) Evaluate the integral found in part (b) to compute $P(X + Y > 1)$.

Problem 5. Let (X, Y) be random variables with a joint probability density function given by

$$f(x, y) = \begin{cases} e^{-(x+y)} & \text{if } x > 0, y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) (5 points) Verify that $f(x, y)$ is a valid probability density function.

(b) (5 points) Compute the probability that $X > 2Y$.

(c) (5 points) Find the CDF of (X, Y).

(d) (5 points) Use your answer in (c) to find $P(X \geq 1 \text{ or } Y \geq 2)$.