Summer 2025

MATH 149A: Probability and Mathematical Statistics

Final Review Problems
Solutions

Midterm review problems

Please review and revisit the problems presented in the Midterm Review. The problems on the Final Exam related to the first half of the course will be similar to those covered in the Midterm Review.

Markov's and Chebyshev's inequalities

1. Let $\mathcal{S}$ be a random variable representing the time (in seconds) it takes for a computer program to execute. The mean execution time is 100 seconds, and the variance is 25 seconds.

   (*a*) Use Markov's inequality to estimate the probability that the program's execution time is at least 5 minutes.

   Solution. We are asked to estimate $P(\mathcal{S} \geq 300)$, since 5 minutes = 300 seconds. Markov's inequality states that

   $$P(\mathcal{S} \geq 300) \leq \frac{\mathbb{E}(\mathcal{S})}{300} = \frac{100}{300} = \frac{1}{3}.$$

   (*b*) Use Chebyshev's inequality to give a lower bound on the probability that the program's execution time is within 15 seconds of the mean.

   Solution. We are asked to estimate $P(|\mathcal{S} - 100| < 15)$, the probability that $\mathcal{S}$ lies within 15 seconds of its mean. Chebyshev's inequality states:

   $$P(|\mathcal{S} - 100| \geq 15) \leq \frac{\mathrm{Var}(\mathcal{S})}{15^2} = \frac{25}{15^2} = \frac{1}{9}.$$

   Therefore, the lower bound on the desired probability is:

   $$P(|\mathcal{S} - 100| < 15) = 1 - P(|\mathcal{S} - 100| \geq 15) \geq 1 - \frac{1}{9} = \frac{8}{9}.$$

Joint and marginal distributions

1. Imagine a zoo with twelve exhibits, each labeled with an animal type and a caretaker. Let X represent the animal type, and Y represent the caretaker of the selected exhibit. The joint probability mass function is given by the table below.

x/y	Mammals	Birds	Reptiles	Amphibians
Caretaker A	0	1/12	1/12	1/12
Caretaker B	3/12	0	0	1/12
Caretaker C	0	1/12	2/12	0
Caretaker D	0	0	0	2/12

   (*a*) What is the probability that a randomly chosen exhibit features an Amphibian cared for by Caretaker C?

   Solution. The probability is given by $P(X = \text{Amphibians}, Y = \text{Caretaker C}) = 0$.

(b) Compute the marginal PMF of X.

Solution. The marginal PMF of X is obtained by summing the probabilities over all values of Y for each fixed value of X:

$$\begin{aligned}
P(X = \text{Mammals}) &= 0 + \frac{3}{12} + 0 + 0 = \frac{1}{4},\\
P(X = \text{Birds}) &= \frac{1}{12} + 0 + \frac{1}{12} + 0 = \frac{1}{6},\\
P(X = \text{Reptiles}) &= \frac{1}{12} + 0 + \frac{2}{12} + 0 = \frac{1}{4},\\
P(X = \text{Amphibians}) &= \frac{1}{12} + \frac{1}{12} + 0 + \frac{2}{12} = \frac{1}{3}.
\end{aligned}$$

(c) Compute the marginal PMF of Y.

Solution. The marginal PMF of Y is obtained by summing the probabilities over all values of X for each value of Y:

$$\begin{aligned}
P(Y = \text{Caretaker A}) &= 0 + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{1}{4},\\
P(Y = \text{Caretaker B}) &= \frac{3}{12} + 0 + 0 + = \frac{1}{3},\\
P(Y = \text{Caretaker C}) &= 0 + \frac{1}{12} + \frac{2}{12} + 0 = \frac{1}{4},\\
P(Y = \text{Caretaker D}) &= 0 + 0 + 0 + \frac{2}{12} = \frac{1}{6}.
\end{aligned}$$

2. Let X and Y have the joint PDF $f(x, y) = \frac{x}{9} + \frac{y^2}{3}$, $0 < x < 1, 0 < y < 2$, and zero elsewhere.

(a) Check that $f(x, y)$ is indeed a valid PDF.

Solution. To check if $f(x, y)$ is a valid PDF, we need to verify two conditions:

- $f(x, y) \geq 0$ for all x, y.
- $\int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$.

Let's check these conditions:

- For $f(x, y) = \frac{x}{9} + \frac{y^2}{3}$, it is clear that the terms are always non-negative.
- $\int\limits_0^1 \int\limits_0^2 \left(\frac{x}{9} + \frac{y^2}{3}\right) dy\, dx = \int\limits_0^1 \left(\left(\frac{xy}{9} + \frac{y^3}{9}\right)\bigg|_{y=0}^{2}\right) dx = \int\limits_0^1 \frac{2x+8}{9}\, dx = \frac{x^2+8x}{9}\bigg|_0^1 = \dfrac{1+8}{9} = 1$, so the second condition is satisfied.

Therefore, $f(x, y)$ is a valid PDF.

(b) Find the marginal PDF of X.

Solution. The marginal PDF of X is obtained by integrating $f(x, y)$ with respect to y over the entire range of y:

$$f_X(x) = \int\limits_0^2 \left(\frac{x}{9} + \frac{y^2}{3}\right) dy = \left(\frac{xy}{9} + \frac{y^3}{9}\right)\bigg|_{y=0}^{2} = \frac{2x+8}{9}.$$

(c) Find the marginal PDF of Y.

Solution. The marginal PDF of Y is obtained by integrating $f(x, y)$ with respect to x over the entire range of x:

$$f_Y(y) = \int\limits_0^1 \left(\frac{x}{9} + \frac{y^2}{3}\right) dx = \left(\frac{x^2}{18} + \frac{xy^2}{3}\right)\bigg|_{x=0}^{1} = \frac{1}{18} + \frac{y^2}{3}.$$

(*d*) Use your answer in (*c*) to find $P(0.3 < Y < 0.9)$.
Solution. The probability is given by the integral of $f_Y(y)$ over the specified range:

$$P(0.3 < Y < 0.9) = \int_{0.3}^{0.9} \left(\frac{1}{18} + \frac{y^2}{3}\right) dy = \left(\frac{y}{18} + \frac{y^3}{9}\right)\bigg|_{0.3}^{0.9} = \frac{0.9}{18} + \frac{0.9^3}{9} - \frac{0.3}{18} - \frac{0.3^3}{9} \approx 0.111.$$

Expectation of two random variables

1. Consider random variables U and V with ranges $\{0,1,2\}$ and $\{0,1\}$ and joint probability mass function $p(u,v) = \frac{3u+v}{21}$.

(*a*) Check that $p(u,v)$ is indeed a valid PMF.
Solution. To check if $p(u,v)$ is a valid probability mass function, we need to verify two conditions:

- $p(u,v) \geq 0$ for all u, v.
- $\sum_u \sum_v p(u,v) = 1$.

Let's check these conditions:

- For $p(u,v) = \frac{3u+v}{21}$, it is clear that the numerator is always non-negative, and the denominator is positive, so $p(u,v) \geq 0$ for all u, v.
- $\sum_u \sum_v p(u,v) = \sum_{u=0}^{2} \sum_{v=0}^{1} \frac{3u+v}{21} = \frac{0+1+3+4+6+7}{21} = 1$, so the second condition is satisfied.

Therefore, $p(u,v)$ is a valid PMF.

(*b*) Compute $\mathbb{E}(U^2)$ and $\mathbb{E}(UV)$.
Solution. The expected values are given by

$$\mathbb{E}(U^2) = \sum_u \sum_v u^2 \cdot p(u,v) = \sum_{u=0}^{2} \sum_{v=0}^{1} \frac{3u+v}{21} \cdot u^2 = \frac{0+1}{21} \cdot 0^2 + \frac{3+4}{21} \cdot 1^2 + \frac{6+7}{21} \cdot 2^2 = \frac{59}{21},$$

$$\mathbb{E}(UV) = \sum_u \sum_v p(u,v) \cdot uv = \left(\frac{0}{21} + \frac{1}{21} + \frac{3}{21} + \frac{6}{21}\right) \cdot 0 + \frac{4}{21} \cdot 1 + \frac{7}{21} \cdot 2 = \frac{18}{21}.$$

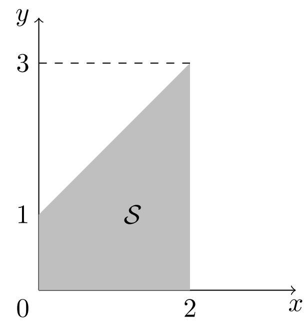
(*c*) Find $\mathbb{E}(4U^2 - 3UV)$.
Solution. Using linearity of expectation, we obtain

$$\mathbb{E}(4U^2 - 3UV) = 4\mathbb{E}(U^2) - 3\mathbb{E}(UV) = 4 \cdot \frac{59}{21} - 3 \cdot \frac{18}{21} = \frac{182}{21} = \frac{26}{3}.$$

2. Consider continuous random variables X and Y with joint probability density function given by:

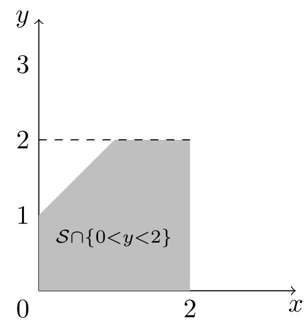
$$f(x,y) = \begin{cases} \frac{3}{40} \cdot (x^2 + y^2) & \text{if } 0 < x < 2,\ 0 < y < x+1 \\ 0 & \text{otherwise.} \end{cases}$$

(*a*) Sketch the region $\mathcal{S} \subset \mathbb{R}^2$ where $f(x,y) \neq 0$.
Solution. The region $\mathcal{S}$ is the set of points (x,y) such that $0 < x < 2$ and $0 < y < x+1$. The region is bounded below by the x-axis, on the left by $x = 0$, on the right by $x = 2$, and above by the line $y = x + 1$:



(b) Find $P(0 < Y < 2)$.

Solution. To find $P(0 < Y < 2)$, we integrate $f(x, y)$ over the subset $\mathcal{S}$, cut out by the additional constraint $y < 2$:



$$P(0 < Y < 2) = \int_0^2 \int_0^{\min(x+1,2)} \frac{3}{40}(x^2 + y^2)\, dy\, dx =$$

$$\int_0^1 \left(\frac{3}{40}x^2 y + \frac{3}{120}y^3 \right)\Bigg|_{y=0}^{x+1} dx + \int_1^2 \left(\frac{3}{40}x^2 y + \frac{3}{120}y^3 \right)\Bigg|_{y=0}^{2} dx =$$

$$\int_0^1 \left(\frac{3}{40}x^2(x+1) + \frac{3}{120}(x+1)^3 \right) dx + \int_1^2 \left(\frac{6}{40}x^2 + \frac{24}{120} \right) dx =$$

$$\frac{3}{40} \int_0^1 \left(x^3 + x^2 + \frac{x^3+1}{3} + x^2 + x \right) dx + 0.05 \int_1^2 \left(3x^2 + 4 \right) dx =$$

$$\frac{3}{40} \left(\frac{x^4}{3} + \frac{2x^3}{3} + \frac{x^2}{2} + \frac{x}{3} \right)\Bigg|_0^1 + 0.05(x^3 + 4x)\Bigg|_1^2 = \frac{11}{80} + \frac{11}{20} = \frac{55}{80} = \frac{11}{16}.$$

(c) Set up the integral expression for computing the expected value of $\sin(1 - XY)$. Do not evaluate it.

Solution. $\mathbb{E}(\sin(1 - XY)) = \frac{3}{40} \int_0^2 \int_0^{x+1} \sin(1 - xy)(x^2 + y^2)\, dy\, dx.$

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