MATH 146B: Ordinary and Partial Differential Equations

Midterm Review Problems

Solutions

Taylor series and radius of convergence

Problem 1. Find the Taylor series centered at x_0 for the provided function. Additionally, determine the radius of convergence of the series.

(a) (1 point) $f(x) = \frac{5}{x^3}$ and $x_0 = -2$.

Solution: the Taylor series centered at -2 for $f(x) = \frac{5}{x^3}$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(-2)}{n!}(x+2)^n$$

We can find the derivatives of $f(x)$ and evaluate them at $x_0 = -2$:

$$f'(x) = -\frac{5 \cdot 3}{x^4}, \quad f''(x) = \frac{5 \cdot 3 \cdot 4}{x^5}, \quad f'''(x) = -\frac{5 \cdot 3 \cdot 4 \cdot 5}{x^6}, \ldots, \quad f^{(n)}(x) = (-1)^n \frac{5 \cdot (n+2)!}{2x^{n+3}}, \ldots$$

Evaluating at $x_0 = -2$, we get

$$f(-2) = \frac{5}{(-2)^3} = -\frac{5}{8}, \quad f'(-2) = -\frac{15}{(-2)^4} = -\frac{15}{16},$$

$$f''(-2) = \frac{60}{(-2)^5} = -\frac{15}{8}, \quad f^{(n)}(-2) = -\frac{5n!}{2^{n+4}}.$$

Therefore, $\frac{f^{(n)}(-2)}{n!} = -\frac{5(n+2)!}{2^{n+4}n!} = -\frac{5(n+2)(n+1)}{2^{n+4}}$ giving the Taylor series

$$f(x) = -5 \sum_{n \geq 0} \frac{(n+2)(n+1)}{2^{n+4}}(x+2)^n.$$

The radius of convergence of this series can be determined as $R = \lim_{n\to\infty} \frac{\frac{5(n+2)(n+1)}{2^{n+4}}}{\frac{5(n+3)(n+2)}{2^{n+5}}} = \lim_{n\to\infty} \frac{2^{n+5}(n+2)(n+1)}{2^{n+4}(n+3)(n+2)} = 2.$

(b) (1 point) $g(x) = x + 3x^2 + e^{-x}$ and $x_0 = 0$.

Solution: the Taylor series centered at x_0 for $g(x) = x + 3x^2 + e^{-x}$ is given by:

$$g(x) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!}x^n$$

We can find the derivatives of $g(x)$ and evaluate them at x_0:

$$g'(x) = 1 + 6x - e^{-x}, \quad g''(x) = 6 + e^{-x}, \quad g'''(x) = -e^{-x}, \ldots, g^{(n)}(x) = (-1)^n e^{-x}, \ldots$$

Evaluating at $x_0 = 0$, we get:

$$g(0) = 1, \quad g'(0) = 1 + 0 - 1 = 0, \quad g''(0) = 6 + 1 = 7, \quad g'''(0) = -1, \ldots, g^{(n)}(0) = (-1)^n, \ldots$$

Therefore, the coefficient $\frac{g^{(n)}(0)}{n!}$ equals:

$$c_0 = \frac{g(0)}{0!} = 1, \quad c_1 = \frac{g'(0)}{1!} = 0, \quad c_2 = \frac{g''(0)}{2!} = \frac{7}{2}, \ldots, \frac{g^{(n)}(0)}{n!} = \frac{(-1)^n}{n!}, \ldots$$

Thus, the Taylor series is:

$$g(x) = 1 + \frac{7x^2}{2} + \sum_{n=3}^{\infty} \frac{(-1)^n}{n!} x^n$$

The radius of convergence of this series is infinite.

Power series solutions near an ordinary point

Problem 2. Consider the equation $y'' + xy' + 2y = 0$ and the point $x_0 = 0$.

(a) (1 point) Seek a solution in the form $y = \sum_{n \geq 0} c_n (x - x_0)^n$ and determine the first three nonzero terms (expressions in the two parameters c_0 and c_1).

Solution: we are aiming to find a solution of the form $y = \sum_{n \geq 0} c_n x^n$ for the given equation. Substituting this into the equation yields:

$$\sum_{n \geq 0} (n+2)(n+1)c_{n+2}x^n + x \sum_{n \geq 0} (n+1)c_{n+1}x^n + 2 \sum_{n \geq 0} c_n x^n = 0.$$

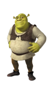
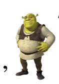
Simplifying this, we get:

$$2c_2 + 2c_0 + \sum_{n \geq 1} ((n+2)(n+1)c_{n+2} + (n+2)c_n)x^n = 0.$$

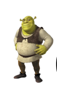
By equating coefficients of x^n to zero for each $n \geq 0$, we can determine expressions for c_n.
For $n = 0$, the equation becomes: $2c_2 + 2c_0 = 0$, resulting in $c_2 = -c_0$.
For $n = 1$, we have: $2c_3 + c_1 = 0$, giving

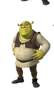
$$c_3 = -\frac{c_1}{2}.$$

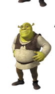
Thus, the first four nonzero terms are $c_0, c_1, c_2 = -c_0$, and $c_3 = -\frac{c_1}{2}$.

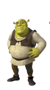
(b) (1 point) Let  denote the solution corresponding to the values $c_0 = 5$ and $c_1 = -3$. Utilize your findings in part (a) to compute the values of (x_0), $'(x_0)$, and $''(x_0)$.

Solution: substituting $c_0 = 5$ and $c_1 = -3$ into the expressions derived in part (a), we get:

$(0) = c_0 = 5,$

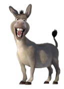
$'(0) = c_1 = -3,$

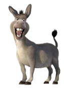
$''(0) = 2c_2 = -2c_0 = -10.$

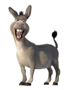
Hence, $(x_0) = 5$, $'(x_0) = -3$, and $''(x_0) = -10$.

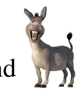
(c) (1 point) Let  denote the solution corresponding to the values $c_0 = -2$ and $c_1 = 4$. Compute the values of (x_0), $'(x_0)$ and $''(x_0)$.

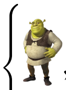
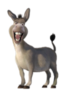
Solution: substituting $c_0 = -2$ and $c_1 = 4$ into the expressions derived in part (a), we get:

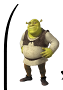
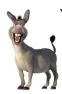
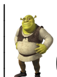
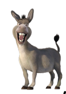
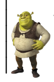
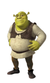
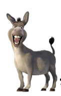
$(0) = c_0 = -2,$

$'(0) = c_1 = 4,$

$''(0) = 2c_2 = -2c_0 = 4.$

Hence, $(x_0) = -2$, $'(x_0) = 4$, and $''(x_0) = 4$.

(d) (1 point) Check if the functions $\{$, $\}$ form a fundamental set of solutions.

Solution: we compute W (, ) $(0) =$ | (0) (0) ; $'(0)$ $'(0)$ | $= \begin{vmatrix} 5 & -2 \\ -3 & 4 \end{vmatrix} = 5 \cdot 4 - (-2) \cdot (-3) = 20 - 6 = 14 \neq 0$, hence,  and , despite being close friends, maintain independence.

Euler equation

Problem 3. Find the general solution for each of the following equations on the interval $(0, a)$, where $a > 0$. (see Lecture 8 notes).

(a) (5 points) $x^2y'' + 7xy' + 9y = 0$.

Solution: to find the general solution for the given equation on the interval $(0, a)$, where $a > 0$, we compute $F(r) = r^2 + (\alpha - 1)r + \beta = r^2 + 6r + 9 = (r + 3)^2$, which has a single root $r_1 = -3$. Consequently, the functions $y_1 = x^{-3}$ and $y_2 = \ell n(x) \cdot x^{-3}$ form a fundamental set of solutions.

Thus, the general solution is given by:

$$y = C_1x^{-3} + C_2\ell n(x)x^{-3}.$$

(b) (5 points) $x^2y'' - xy' - 3y = 0$.

Solution: to find the general solution for the equation $x^2y'' - xy' - 3y = 0$ on the interval $(0, a)$, where $a > 0$, we compute $F(r) = r^2 + (\alpha - 1)r + \beta = r^2 - 2r + 3$, which has two complex roots $r_1 = 1 + 2i$ and $r_2 = 1 - 2i$. Consequently, the functions $y_1 = x\cos(2\ell n(x))$ and $y_2 = x\sin(2\ell n(x))$ form a fundamental set of solutions.

Thus, the general solution is given by

$$y = C_1 x \cos(2\ell n(x)) + C_2 x \sin(2\ell n(x)).$$

(c) (5 points) $x^2 y'' - 5xy' + 8y = 0$.

Solution: to find the general solution for the equation $x^2 y'' - 5xy' + 8y = 0$ on the interval $(0, a)$, where $a > 0$, we compute $F(r) = r^2 + (\alpha - 1)r + \beta = r^2 - 6r + 8$, which has two real roots $r_1 = 2$ and $r_2 = 4$. Consequently, the functions $y_1 = x^2$ and $y_2 = x^4$ form a fundamental set of solutions.

Thus, the general solution is given by $y = C_1 x^2 + C_2 x^4$.

Singularities

Problem 4. For each equation, provide a complete list of singular points and determine their types.

(a) (1 point) $(5x^2 + 3)(x - 4)(x + 3)^2 y'' + (x^2 - 16)y' + (x + 3)y = 0$.

Solution: the singular points are the real zeros of the polynomial $(5x^2 + 3)(x - 4)(x + 3)^2$, which are $x_0 = 4$ and $x_1 = -3$. We evaluate the limits:

$$\lim_{x \to 4} (x - 4) \cdot \frac{(x^2 - 16)}{(5x^2 + 3)(x - 4)(x + 3)^2} = 0$$

and

$$\lim_{x \to 4} (x - 4)^2 \cdot \frac{(x + 3)}{(5x^2 + 3)(x - 4)(x + 3)^2} = 0.$$

Both functions $(x - 4) \cdot \frac{(x^2 - 16)}{(5x^2 + 3)(x - 4)(x + 3)^2}$ and $(x - 4)^2 \cdot \frac{(x + 3)}{(5x^2 + 3)(x - 4)(x + 3)^2}$ are rational and thus analytic. Hence, the singularity at $x_0 = 4$ is regular.

For $x_1 = -3$, we have

$$\lim_{x \to -3} (x + 3) \cdot \frac{(x^2 - 16)}{(5x^2 + 3)(x - 4)(x + 3)^2} = \lim_{x \to -3} \frac{(x + 4)}{(5x^2 + 3)(x + 3)} = \lim_{x \to -3} \frac{1}{48(x + 3)},$$

which does not exist, indicating that the singularity at $x_1 = -3$ is irregular.

(b) (1 point) $(x^2 + x^4)y'' + 5x \sin(x) y' + 2024y = 0$.

Solution: the singular points are determined by the real roots of the polynomial equation $x^2 + x^4 = x^2(x^2 + 1)$, which simplifies to $x_0 = 0$. We proceed similarly to the previous problem.

For $x_0 = 0$, we evaluate the limits:

$$\lim_{x \to 0} \frac{5x \sin(x)}{x^2 + x^4} = \lim_{x \to 0} \frac{5 \sin(x)}{x + x^3} = \lim_{x \to 0} \frac{5 \sin(x)}{x} = 5.$$

and

$$\lim_{x \to 0} \frac{2024x^2}{x^2 + x^4} = 2024.$$

Both functions $\frac{5x \sin(x)}{x^2 + x^4}$ and $\frac{2024x^2}{x^2 + x^4}$ are analytic (product of two analytic functions is analytic). Therefore, the singularity at $x_0 = 0$ is regular.