Lectures on Ordinary and Partial Differential Equations

Contents

Lecture 0

Exploring Mathematics: Zeta Functions, Game Theory, and Beyond

Note: This material is complementary and optional.

In this session, we will discuss intriguing mathematical results and concepts through a game. Participants were asked to name a famous mathematician and one of their significant results.

1. Riemann Zeta Function and Euler's Product Formula

The Basel Problem introduces us to the expression for the Riemann Zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = (-1)^{\frac{s}{2}+1} \frac{(2\pi)^s B_s}{2s!},$$

where $s = 2$ results in the well-known formula:

$$\zeta(2) = \frac{\pi^2}{6}.$$

For positive even integers s, the expression generalizes as:

$$\zeta(s) = (-1)^{\frac{s}{2}+1} \frac{(2\pi)^s B_s}{2s!},$$

where B_s denotes the s^{th} Bernoulli number.

Euler's Product Formula

Now, let's explore Euler's product formula for $s = 1$:

$$\zeta(1) = \prod_{p \text{ prime}} \left(1 - p^{-1}\right),$$

a surprising representation involving the prime numbers.

This result generalizes to:

$$\zeta(s) = Y \prod_{p \text{ prime}} \left(1 - p^{-s}\right),$$

and it uncovers a profound connection between the Riemann Zeta function and prime numbers.

2. Nash Equilibrium and the Prisoner's Dilemma

Nash Equilibrium is a key concept in game theory, defined in scenarios involving n players, each with a finite set of strategies and utility functions.

Definition. A set of strategies is a **Nash Equilibrium** if, when each player knows the strategies of the others, no player has an incentive to change their strategy.

To illustrate, consider the classic example of the *Prisoner's Dilemma*. Two suspects are arrested, and each has the option to cooperate (C) or betray (B) the other. The outcomes, in terms of years in prison, are as follows:

Prisoner 1/Prisoner 2	Cooperate (C)	Betray (B)
Cooperate (C)	(-1, -1) years	(-3, 0) years
Betray (B)	(0, -3) years	(-2, -2) years

Remark. In the scenario of the Prisoner's Dilemma, if both players choose to cooperate, they would each receive a better payoff (1 instead of 2 years imprisonment). Furthermore, the collectively payoff (-2 years) is better than for any other combination of choices. However, this situation is not a Nash equilibrium because, given their individual incentives, each player has an incentive to betray the other to minimize their own individual sentence, leading to the dilemma's characteristic outcome. Nash equilibrium is reached when no player can unilaterally change their strategy to achieve a better outcome, and in the Prisoner's Dilemma, betraying is a dominant strategy for each player.

Finding Nash Equilibria involves identifying strategies where no player has an incentive to unilaterally deviate. In the Prisoner's Dilemma, one Nash Equilibrium is when both betray (B, B). Here, if one changes strategy, they would end up with a worse outcome.

Recommendation: I highly recommend watching the movie *A Beautiful Mind*, which explores Nash Equilibria and game theory in an engaging manner.

"A Beautiful Mind" provides insights into the life of John Nash, a brilliant mathematician who made significant contributions to game theory, including the concept of Nash Equilibrium. The movie depicts how Nash's theories impacted various aspects of decision-making.

3. Pythagorean Theorem and Congruent Numbers

The Pythagorean Theorem states that in a right triangle, the square of the length of the hypotenuse (c) is equal to the sum of the squares of the lengths of the other two sides (a and b):

$$c^2 = a^2 + b^2.$$

This fundamental theorem, attributed to the ancient Greek mathematician Pythagoras, has applications in geometry and trigonometry.

Congruent Numbers

The Pythagorean Theorem is intimately connected to the concept of congruent numbers. A positive integer n is a congruent number if there exists a right-angled triangle with rational side lengths such that its area is equal to n. In other words, n is congruent if and only if there exist rational numbers a, b, and c satisfying:

$$a^2 + b^2 = c^2 \quad \text{and} \quad \frac{ab}{2} = \frac{n}{2}.$$

The first few congruent numbers are

$$\begin{gathered}5, 6, 7, 13, 14, 15, 20, 21, 22, 23, 24, 28, 29, 30, 31, 34, 37, 38,\\ 39, 41, 45, 46, 47, 52, 53, 54, 55, 56, 60, 61, 62, 63, 65, 69, 70,\\ 71, 77, 78, 79, 80, 84, 85, 86, 87, 88, 92, 93, 94, 95, 96, 101, 102,\\ 103, 109, 110, 111, 112, 116, 117, 118, 119, 120.\end{gathered}$$

For example, consider the right triangle with sides $3, 4$, and 5. The area of this triangle is 6. Another example is the triangle with side lengths $20/3, 3/2$, and $41/6$. Its area is equal to 5.

The study of congruent numbers involves deep connections between number theory and geometry, providing a fascinating intersection between these mathematical domains.

4. Euclid's Result on Perpendicular Lines

Euclid, often referred to as the "father of geometry," made significant contributions to the understanding of geometric concepts. One of his notable results involves the nature of perpendicular lines.

One of Euclid's Propositions states that, in a plane, from any point not on a given line, there is exactly one line perpendicular to the given line. This proposition lays the foundation for understanding the unique relationship between points and lines in Euclidean geometry.

Euclid and His Modern Rivals

Euclid's work, particularly his compilation "Elements," is a collection of books that has had a profound and enduring impact on the study of geometry. "Elements" covers topics from plane geometry to number theory.

The novel "Euclid and His Modern Rivals" by Lewis Carroll, known for his "Alice's Adventures in Wonderland," provides a fascinating exploration of the debates and controversies surrounding Euclid's work. Carroll, a mathematician himself, wrote the book as a satirical response to the reformist efforts in mathematical education during his time. The novel delves into the clash between traditional Euclidean geometry and emerging mathematical perspectives, offering a unique blend of mathematical history and literary storytelling.

5. Poincaré and the Three-Body Problem

The three-body problem is a classic conundrum in celestial mechanics that explores the motion of three celestial bodies under the influence of their mutual gravitational attraction. While the two-body problem can be solved analytically, the addition of a third body renders the system chaotic and analytically intractable.

This problem can be traced back to the works of Henri Poincaré, who laid the foundation for the study of chaotic systems. The gravitational interactions between three celestial bodies create intricate and unpredictable trajectories, defying straightforward analytical solutions. Poincaré's insights into chaotic dynamics paved the way for the development of chaos theory in modern mathematics.

Formulating the Three-Body Problem

Consider three celestial bodies, each with mass m_1, m_2, and m_3, respectively, in space. The interaction between any two bodies is given by Newton's law of gravitation:

$$F_{ij} = G \cdot \frac{m_i \cdot m_j}{r_{ij}^2},$$

where F_{ij} is the gravitational force between bodies i and j, G is the gravitational constant, and r_{ij} is the distance between bodies i and j.

The equations of motion for each body in a three-body system can be expressed using Newton's second law:

$$F_i = m_i \cdot a_i,$$

where F_i is the net force acting on body i and a_i is its acceleration.

Solving the system of differential equations to predict the positions (x_i) and velocities $(\dot{x}_i)$ of the three bodies as functions of time

$$\begin{cases} m_1\ddot{x}_1 &= G\frac{m_1 m_2}{r_{12}^3}(x_2 - x_1) + G\frac{m_1 m_3}{r_{13}^3}(x_3 - x_1), \\ m_2\ddot{x}_2 &= G\frac{m_2 m_1}{r_{21}^3}(x_1 - x_2) + G\frac{m_2 m_3}{r_{23}^3}(x_3 - x_2), \\ m_3\ddot{x}_3 &= G\frac{m_3 m_1}{r_{31}^3}(x_1 - x_3) + G\frac{m_3 m_2}{r_{32}^3}(x_2 - x_3), \end{cases}$$

poses a difficult challenge due to the chaotic nature of the interactions. This challenge forms the essence of the three-body problem.

The study of the three-body problem has applications in various fields, from astrodynamics to physics and mathematics. Understanding the complexities of such systems is crucial for predicting the behavior of celestial bodies in gravitational systems.

For those intrigued by this fascinating topic, exploring Poincaré's contributions and the subsequent developments in chaos theory provides a deeper appreciation for the intricate dance of celestial bodies in our universe.

Conclusion

These topics offer a glimpse into the diverse and captivating world of mathematics. Feel free to explore further or watch relevant movies to enhance your understanding.

Lecture 1
Linear Equations of Order n

An n^{th}-order linear differential equation is an equation of the form

$$P_n(x)\frac{d^n y}{dx^n} + P_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_1(x)\frac{dy}{dx} + P_0(x)y = G(x).$$

We assume that the functions $P_0, \ldots, P_n$, and G are continuous real-valued functions on some open interval $(a < x < b)$ and that P_n is nowhere zero in this interval. Then, dividing by P_n, we rewrite the equation in the form

$$\frac{d^n y}{dx^n} + p_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_1(x)\frac{dy}{dx} + p_0(x)y = g(x).$$

In order to work with linear differential equations, it is helpful to introduce differential operators.

Definition. A **linear differential operator of order** n is a mapping from the space $\mathcal{C}^m(a,b)$ of functions (with at least m continuous derivatives) on the open interval (a,b) to itself, given by an expression

$$\mathcal{D} = \sum_{j=0}^{n} g_j(x)\frac{d^j}{dx^j},$$

where f_j are continuous functions on the interval.

Example.

1. Consider the first-order differential operator $\mathcal{D} = 5\frac{d}{dx}$. Let's evaluate D on the function $f(x) = e^{x^2}$:

$$\mathcal{D}(f(x)) = 5\frac{d(e^{x^2})}{dx} = 10xe^{x^2}.$$

2. Now, let's explore a second-order differential operator $L = \cos(x)\frac{d^2}{dx^2} + 2\frac{d}{dx} - 3$. Evaluating $\mathcal{D}$ on the function $f(x) = \sin(x)$ gives

$$\mathcal{D}(f(x)) = \cos(x)\frac{d^2(\sin(x))}{dx^2} + 2\frac{d(\sin(x))}{dx} - 3\cdot\sin(x) = -\cos(x)\sin(x) + 2\cos(x) - 3\sin(x).$$

3. Lastly, consider a third-order differential operator $\mathcal{D} = \frac{d^3}{dx^3} - 3\frac{d^2}{dx^2} + 2\frac{d}{dx}$. Then

$$\mathcal{D}(x^2) = \frac{d^3(x^2)}{dx^3} - 3\frac{d^2(x^2)}{dx^2} + 2\frac{d(x^2)}{dx} = 0 - 6 + 4x = 4x - 6.$$

We associate the linear differential operator $\mathcal{D} := \frac{d^n}{dx^n} + p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_1(x)\frac{dy}{dx} + p_0(x)$ of order n to the equation $\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}(x)\frac{dy}{dx} + p_n(x)y = F(x)$. The equation can be rewritten as $\mathcal{D}(y) = 0$.

Theorem. Let x_0 be a point in (a,b) with $k_0, k_1, \ldots, k_{n-1}$ arbitrary real numbers. Then the initial value problem

$$\frac{d^n y}{dx^n} + p_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_1(x)\frac{dy}{dx} + p_0(x)y = F(x), \quad y(x_0) = k_0,\, y'(x_0) = k_1, \ldots, y^{(n-1)}(x_0) = k_{n-1}$$

has a unique solution on (a,b).

The equation is said to be *homogeneous* if $F(x) = 0$ and nonhomogeneous otherwise. Since $y = 0$ is a solution of any homogeneous equation, we call it the *trivial solution.* Any other solution is nontrivial.

If $y_1, y_2, \ldots, y_n$ are defined on (a, b) and $c_1, c_2, \ldots, c_n$ are constants, then

$$y = c_1y_1 + c_2y_2 + \ldots + c_ny_n$$

is a *linear combination* of $\{y_1, y_2, \ldots, y_n\}$. It is straightforward to check that if $y_1, y_2, \ldots, y_n$ are solutions on (a, b), then any linear combination of $\{y_1, y_2, \ldots, y_n\}$ is also a solution.

We call $\{y_1, y_2, \ldots, y_n\}$ a *fundamental set of solutions* on (a, b) if every solution on (a, b) can be expressed as a linear combination of $\{y_1, y_2, \ldots, y_n\}$. In this case, we say that

$$y = c_1y_1 + c_2y_2 + \ldots + c_ny_n$$

with arbitrary real numbers $c_1, c_2, \ldots, c_n$ is the *general solution* on (a, b).

Definition. The functions $y_1, y_2, \ldots, y_n$ are **linearly dependent** on an interval (a, b) if there exist numbers $c_1, c_2, \ldots, c_n$ such that the linear combination $c_1y_1 + c_2y_2 + \ldots + c_ny_n$ is identically equal to zero on the entire interval (a, b), i.e., $(c_1y_1 + c_2y_2 + \ldots + c_ny_n)(x) = 0$ for all $a < x < b$. Otherwise, we say that $y_1, y_2, \ldots, y_n$ are **linearly independent** on that interval.

Theorem. A set $\{y_1, y_2, \ldots, y_n\}$ of n solutions on (a, b) is a fundamental set if and only if it is linearly independent on (a, b).

This result naturally leads us to a question.

Question. Given a collection of functions $y_1, y_2, \ldots, y_n$, how can we determine if they are linearly independent on an interval (a, b)?

Wronskian and Linear Independence of Functions

The Wronskian of functions $y_1, y_2, \ldots, y_n$ over an interval (a, b) is computed by organizing these functions along with their successive derivatives into a matrix. In this matrix, each row corresponds to a distinct function, while each column corresponds to a different order of the derivative. The Wronskian is subsequently determined as the determinant of this matrix:

$$W(y_1, y_2, \ldots, y_n) := \det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

If the Wronskian of a set of functions is not always zero on (a, b), then these functions are linearly independent.

Example.

1. Let's consider the functions $y_1 = 5, y_2 = 3x - 1,$ and $y_3 = 5x^2 - 2x - 7$.

   The matrix of derivatives is as follows:

   $$\begin{pmatrix} 5 & 3x-1 & 5x^2-2x-7 \\ 0 & 3 & 10x-2 \\ 0 & 0 & 10 \end{pmatrix}$$

   and has the Wronskian (determinant) $W = 5 \cdot 3 \cdot 10 = 150$. Since this Wronskian is a nonzero constant, we can conclude that y_1, y_2, and y_3 are linearly independent on any interval.

2. The Wronskian of functions $f_1(x) = \sin(x)$, $f_2(x) = 5 - 2x$, and $f_3(x) = e^{7x}$ is given by the determinant of the matrix

   $$A = \begin{pmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{pmatrix}.$$

We compute the required derivatives

$$f_1' = \cos(x), \quad f_2' = -2, \quad f_3' = 7e^{7x}$$

$$f_1'' = -\sin(x), \quad f_2'' = 0, \quad f_3'' = 49e^{7x}$$

to get

$$A = \begin{pmatrix} \sin(x) & 5-2x & e^{7x} \\ \cos(x) & -2 & 7e^{7x} \\ -\sin(x) & 0 & 49e^{7x} \end{pmatrix}.$$

Expanding along the third row, we find

$$\begin{aligned} W = \det(A) &= (-1)^{3+1} \cdot (-\sin(x)) \cdot \det \begin{pmatrix} 5-2x & e^{7x} \\ -2 & 7e^{7x} \end{pmatrix} + (-1)^{3+3} \cdot 49e^{7x} \cdot \det \begin{pmatrix} \sin(x) & 5-2x \\ \cos(x) & -2 \end{pmatrix} \\ &= -\sin(x) \cdot (7(5-2x)e^{7x} + 2e^{7x}) + 49e^{7x} \cdot (-2\sin(x) - (5-2x)\cos(x)). \end{aligned}$$

For instance, we can use this result to assess the linear independence of functions f_1, f_2, f_3 over the interval $(-1, 1)$. By calculating $W(0) = 0 + 49(0-5) = -245 \neq 0$, we can conclude their independence.

Theorem. Let $\{y_1, y_2, \ldots, y_n\}$ be solutions of the homogeneous equation $\frac{d^n y}{dx^n} + p_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_1(x)\frac{dy}{dx} + p_0(x)y = 0$ on (a, b). If $W(y_1, y_2, \ldots, y_n)(x_0) \neq 0$ for at least one point $x_0 \in (a, b)$, then the functions $\{y_1, y_2, \ldots, y_n\}$ form a fundamental set of solutions.

Remark. It is crucial to highlight that, for arbitrary functions $y_1, y_2, \ldots, y_n$, the Wronskian does not serve as a criterion for independence. To illustrate this point, consider the functions $y_1(x) = x|x|$ and $y_2(x) = x^2$. Despite the Wronskian being identically zero:

$$W = \det \begin{pmatrix} x|x| & x^2 \\ 2|x| & 2x \end{pmatrix} = 2x^2|x| - 2x^2|x| = 0,$$

these functions are linearly independent on any interval (a, b) containing zero. Take any $\varepsilon \neq 0$ with both $-\varepsilon$ and ε on the interval. Then, for $c_1 x|x| + c_2 x^2 \equiv 0$, we deduce:

$$\begin{cases} c_1\varepsilon^2 + c_2\varepsilon^2 = 0 \\ -c_1\varepsilon^2 + c_2\varepsilon^2 = 0 \end{cases} \Leftrightarrow \begin{cases} c_1 = -c_2 \\ c_1 = c_2 \end{cases},$$

implying $c_1 = c_2 = 0$.

Lecture 2

Linear Equations of Order n with Constant Coefficients

If $a_0, a_1, \ldots, a_n$ are constants with $a_0 \neq 0$, then the differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = F(x)$$

is referred to as a *constant coefficient equation.* We assume that $a_n \neq 0$, so the theorems from the previous section apply with $(a, b) = (-\infty, \infty)$.

If we consider $y = e^{rx}$, then $y' = re^{rx}$, $y'' = r^2 e^{rx}$, and so on, giving

$$\mathcal{D}(e^{rx}) = a_n r^n e^{rx} + \ldots + a_{n-1} r e^{rx} + a_0 e^{rx} = e^{rx}(a_n r^n + \ldots + a_1 r + a_0).$$

We define the characteristic polynomial

$$p(r) = a_0 r^n + a_1 r^{n-1} + \ldots + a_n$$

for the corresponding equation. The solutions of the equation are determined by the zeros of the characteristic polynomial, but there are many possible cases to consider.

Distinct real roots

In the simplest case, where p has n distinct real roots $r_1, r_2, \ldots, r_n$, we obtain n solutions

$$y_1 = e^{r_1 x}, \quad y_2 = e^{r_2 x}, \quad \ldots, \quad y_n = e^{r_n x}.$$

It can be demonstrated that the Wronskian of $\{e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}\}$ is nonzero if $r_1, r_2, \ldots, r_n$ are distinct; hence, $\{e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}\}$ forms a fundamental set of solutions in this case.

Example.

1. To determine the general solution of the differential equation $y''' - 6y'' + 3y' + 10y$, we begin by identifying its characteristic polynomial. Notably, the characteristic polynomial factors as

$$r^3 - 6r^2 + 3r + 10 = (r+1)(r-2)(r-5).$$

   The presence of three distinct real roots, namely $r_1 = -1$, $r_2 = 2$, and $r_3 = 5$, leads to the general solution

$$y = C_1 e^{-x} + C_2 e^{2x} + C_3 e^{5x},$$

   where C_1, C_2, and C_3 are constants.

2. Given that the polynomial $t^4 - 5t^3 - 19t^2 + 29t + 42$ factors as $t^4 - 5t^3 - 19t^2 + 29t + 42 = (t-2)(t+3)(t+1)(t-7)$, find the general solution of the homogeneous differential equation $y^{(4)} - 5y''' - 19y'' + 29y' + 42y = 0$.

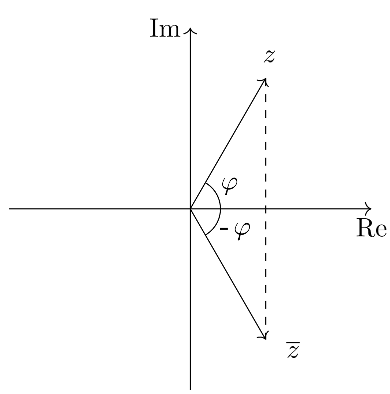
   The characteristic polynomial for the given equation is $p(r) = r^4 - 5r^3 - 19r^2 + 29r + 42 = (r-2)(r+3)(r+1)(r-7)$ and has roots $r_1 = 2, r_2 = -3, r_3 = -1$ and $r_4 = 7$. The general solution is

$$y = C_1 e^{2x} + C_2 e^{-3x} + C_3 e^{-x} + C_4 e^{7x}.$$

As with any other polynomial, the roots of the characteristic polynomial resulting from our ODE can be complex numbers. In the case where the polynomial has real coefficients, the complex roots always appear in pairs: the number $a + bi$ and its conjugate $\overline{a + bi} = a - bi$. This follows from the fact that a real number is equal to its conjugate, hence $p(\overline{z}) = \overline{p(z)} = 0$. Specifically, the number of complex roots of a polynomial with real coefficients is always even.

An interesting observation is that while a polynomial of degree n may have no real roots at all ($z^2 = -1, z^{100} = -5.7$, etc.), it always has n complex roots, counted with multiplicities. This statement is known as the fundamental theorem of algebra and has many interesting proofs, originating in topology, complex analysis, Galois theory...

Every complex number $z = a + bi$ can be expressed as $r(\cos(\varphi) + i\sin(\varphi))$, where $r = \sqrt{a^2 + b^2}$ is the absolute value and φ is the angle formed with the positive x-axis.



This expression is called the *polar form* of a complex number, with r being the *magnitude* and φ being the *argument*. When we multiply two complex numbers, the magnitudes get multiplied, and the arguments are added up. In particular, the set of complex numbers with magnitude 1 forms the unit circle and is closed under multiplication; the product of two such numbers has a magnitude of 1 as well.

Example. Let $z = \cos(\alpha) + i\sin(\alpha)$ and $w = \cos(\beta) + i\sin(\beta)$ represent two complex numbers. Notably, both z and w lie on the unit circle, denoted by $|z| = |w| = 1$, forming angles a and β with the positive x-axis, respectively.

Consider the product zw. We observe that $|zw| = |z| \cdot |w| = 1 \cdot 1 = 1$, and the argument of zw is given by $Arg(zw) = Arg(z) + Arg(w) = \alpha + \beta$. This leads to the conclusion:

$$zw = \cos(\alpha + \beta) + i\sin(\alpha + \beta).$$

Alternatively, we can expand the expression for zw as:

$$zw = (\cos(\alpha) + i\sin(\alpha))(\cos(\beta) + i\sin(\beta)) = \cos(\alpha)\cos(\beta) + i\cos(\alpha)\sin(\beta) + i\sin(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta).$$

Equating real and imaginary parts, we obtain the familiar formulas for the cosine and sine of the sum of two angles:

- $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
- $\sin(\alpha + \beta) = \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta)$.

Now, let's explore some examples of solutions to equations involving complex numbers.

The polynomial $z^n = 1$ has 1 or 2 roots in the real numbers, but it has n complex roots.

To understand the complex roots, we use the polar form of complex numbers. For any complex number $z = r(\cos(\varphi) + i\sin(\varphi))$, we have $|z| = r$ and $\mathrm{Arg}(z) = \varphi$.

For the equation $z^n = 1$, we explore its solutions in the complex plane. This equation implies that the magnitude of z is equal to 1, indicating that all solutions lie on the unit circle.

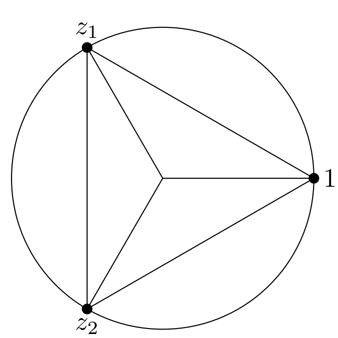
The argument of z satisfies the relation $n\mathrm{Arg}(z) = 2\pi k$ for some integer k. This condition results in a set of possible arguments given by $\mathrm{Arg}(z) \in \left\{\frac{2\pi k}{n} \mid k \in \mathbb{Z}\right\}$.

As $\mathrm{Arg}(z)$ is defined within the range $0 \leq \mathrm{Arg}(z) < 2\pi$, we further refine the set of solutions to $\mathrm{Arg}(z) \in \left\{\frac{2\pi k}{n} \mid 0 \leq k \leq n-1\right\}$. In simpler terms, the solutions form a sequence of points on the unit circle, equally spaced at intervals of $\frac{2\pi}{n}$ radians. This sequence starts with the point $(1, 0)$, representing the first solution.

Let's consider examples for $n = 3$ and $n = 5$.

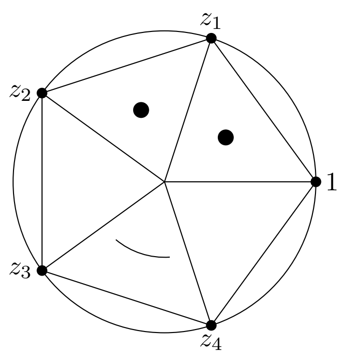
Example. 1. $z^3 = 1$ The roots are given by

$$\begin{aligned}
z_0 &= \cos\left(\frac{2\pi \cdot 0}{3}\right) + i\sin\left(\frac{2\pi \cdot 0}{3}\right) = 1 \\
z_1 &= \cos\left(\frac{2\pi \cdot 1}{3}\right) + i\sin\left(\frac{2\pi \cdot 1}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2} \cdot i \\
z_2 &= \cos\left(\frac{2\pi \cdot 2}{3}\right) + i\sin\left(\frac{2\pi \cdot 2}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2} \cdot i
\end{aligned}$$



2. $z^5 = 1$ The roots are given by

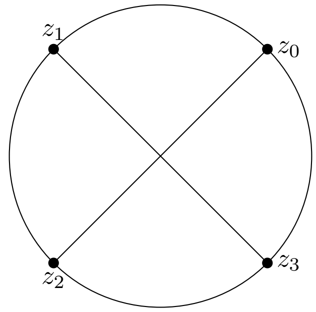
$$z_0 = \cos\left(\frac{2\pi \cdot 0}{5}\right) + i\sin\left(\frac{2\pi \cdot 0}{5}\right) = 1$$
$$z_1 = \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)$$
$$z_2 = \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)$$
$$z_3 = \cos\left(\frac{6\pi}{5}\right) + i\sin\left(\frac{6\pi}{5}\right)$$
$$z_4 = \cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right)$$



Finally, let's solve a slightly different equation. To find the roots of $z^4 = -1$, we look for z such that $|z| = 1$ and $4\text{Arg}(z) = (2k+1)\pi$ for some integer k. The latter equality is equivalent to $\text{Arg}(z) \in \left\{\frac{(2k+1)\pi}{4} \mid 0 \le k \le 3\right\} = \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$.

The solutions are

$$z_0 = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}(1+i)$$
$$z_1 = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}(-1+i)$$
$$z_2 = \cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right) = \frac{1}{\sqrt{2}}(-1-i)$$
$$z_3 = \cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right) = \frac{1}{\sqrt{2}}(1-i).$$



Now, let's resume the exploration of linear ordinary differential equations with constant coefficients.

Distinct complex roots

If the characteristic equation has complex roots, they must occur in conjugate pairs, $\alpha \pm i\beta$, since the coefficients $a_0, \ldots, a_n$ are real numbers. Provided that none of the roots is repeated, the general solution of the differential equation is of the same form as before. Any pair of solutions $e^{(\alpha+i\beta)x}$ and $e^{(\alpha-i\beta)x}$ corresponding to conjugate complex roots $\alpha + i\beta$ and $\alpha - i\beta$ can be replaced by the real-valued solutions $e^{\alpha x}\cos(\beta x) = \frac{e^{(\alpha+i\beta)x}+e^{(\alpha-i\beta)x}}{2}$ and $e^{\alpha x}\sin(\beta x) = \frac{e^{(\alpha+i\beta)x}-e^{(\alpha-i\beta)x}}{2i}$, obtained as the real and imaginary parts of $e^{(\alpha+i\beta)x}$.

Example. Given that the polynomial $t^4+5t^3-5t^2+45t-126$ factors as $t^4+5t^3-5t^2+45t-126 = (t^2+9)(t-2)(t+7)$, find the general solution of the homogeneous differential equation $y^{(4)}+5y'''-5y''+45y'-126y = 0$.

The characteristic polynomial for the given equation is $p(r) = r^4 + 5r^3 - 5r^2 + 45r - 126 = (r+3i)(r-3i)(r-2)(r+7)$ and has two complex roots $r_1 = 3i, r_2 = -3i$ and two real roots $r_3 = 2, r_4 = -7$. The general solution is

$$y = C_1 \cos 3x + C_2 \sin 3x + C_3 e^{2x} + C_4 e^{-7x}.$$

Repeated roots

If the roots of the characteristic equation are not distinct, meaning that some of the roots are repeated, the solution is not the general solution of our differential equation. Recall that for a repeated root r_1 in a second-order linear equation $a_2y'' + a_1y' + a_0y = 0$, two linearly independent solutions are e^{r_1x} and xe^{r_1x}. For an equation of order n, if a root of $p(r) = 0$, say $r = r_1$, has multiplicity s (where $s \leq n$), then $e^{r_1}, e^{r_1x}, te^{r_1x}, x^2e^{r_1x}, \ldots, x^{s-1}e^{r_1x}$ are corresponding solutions.

If a complex root $\alpha + i\beta$ is repeated s times, the complex conjugate $\alpha - i\beta$ is also repeated s times. Corresponding to these $2s$ complex-valued solutions, we can find $2s$ real-valued solutions by noting that the real and imaginary parts of $e^{\alpha+i\beta}x, xe^{\alpha+i\beta}x, \ldots, x^{s-1}e^{\alpha+i\beta}x$ are also linearly independent solutions: $e^{\alpha x}\cos(\beta x), e^{\alpha x}\sin(\beta x), xe^{\alpha x}\cos(\beta x), xe^{\alpha x}\sin(\beta x), \ldots, x^{s-1}e^{\alpha x}\cos(\beta x), x^{s-1}e^{\alpha x}\sin(\beta x)$.

Hence, the general solution of the equation can always be expressed as a linear combination of n real-valued solutions. Consider the following example.

Example. Consider the differential equation $y^{(8)} - 8y^{(4)} + 16y = 0$. Using the characteristic polynomial, we have $r^8 - 8r^4 + 16 = (r^4-4)^2$, and the characteristic equation $(r^4-4)^2 = 0$ gives $r^4 = 4 \Leftrightarrow r^2 = 2$ or $r^2 = -2$. Hence, the roots are $r_1 = \sqrt{2}$, $r_2 = -\sqrt{2}$, $r_3 = \sqrt{2}i$, and $r_4 = -\sqrt{2}i$, each with multiplicity two. The general solution is

$$y = e^{\sqrt{2}x}(C_1 + C_2x) + e^{-\sqrt{2}x}(C_3 + C_4x) + C_5\cos(\sqrt{2}x) + C_6\sin(\sqrt{2}x) + C_7x\cos(\sqrt{2}x) + C_8x\sin(\sqrt{2}x).$$

Lecture 3
The Method of Undetermined Coefficients

The typical approach to solving a nonhomogeneous equation $\mathcal{D}(y) = F(x)$ involves the following sequential strategy.

Step 1. Solve the complementary homogeneous equation $\mathcal{D}(y) = 0$ to find the complementary solution y_c.

Step 2. Find a particular solution y_p of the equation $\mathcal{D}(y) = F(x)$.

Step 3. The general solution is then $y = y_c + y_p$.

In previous lectures, we discussed how to complete Step 1 if $\mathcal{D}$ has constant coefficients. This week, our focus shifts to Step 2.

A particular solution of the nonhomogeneous nth order linear equation with constant coefficients can be obtained by the method of undetermined coefficients, provided that the force function $F(x)$ is of an *appropriate* form. Although the method of undetermined coefficients is not as general as the method of variation of parameters described in the next section, it is usually much easier to use when it is applicable. For instance, if a constant coefficient linear differential operator $\mathcal{D}$ is applied to a polynomial function

$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, an exponential function $e^{\alpha x}$, a sine function $\sin(\beta x)$, or a cosine function $\cos(\beta x)$, the result is a polynomial, an exponential function, or a linear combination of sine and cosine functions, respectively. Hence, if $F(x)$ is a sum of polynomials, exponentials, sines, and cosines, we can expect that it is possible to find $y(x)$ by choosing a suitable combination of polynomials, exponentials, and so forth, multiplied by a number of undetermined constants. The constants are then determined by substituting the assumed expression into the equation under consideration.

Example. Consider the differential equation $y'' + y' = -8\cos(2x) + 6\sin(2x)$.

Assume a particular solution of the form $y_p = A\cos(2x) + B\sin(2x)$.

Find first and second order derivatives

$$y_p' = -2A\sin(2x) + 2B\cos(2x) \text{ and } y_p'' = -4A\cos(2x) - 4B\sin(2x).$$

Substitute y_p, y_p', and y_p'' into the differential equation:

$$-4A\cos(2x) - 4B\sin(2x) - 2A\sin(2x) + 2B\cos(2x) = -8\cos(2x) + 6\sin(2x)$$

Equating coefficients of $\cos(2x)$ and $\sin(2x)$, we get the following system of equations:

$$\begin{cases} -4A + 2B = -8 \\ -4B - 2A = 6 \end{cases} \Leftrightarrow \begin{cases} -2A + B = -4 \\ -2B - A = 3. \end{cases}$$

Solving this system will give us the values of $A = 1$ and $B = -2$.

The particular solution is

$$y_p = \cos(2x) - 2\sin(2x).$$

Next we solve the complementary equation $y'' + y' = 0$.

- Characteristic polynomial: $r^2 + r = r(r+1)$.
- Roots: $r_1 = 0$ and $r_2 = -1$.
- General solution: $y_c = C_1 + C_2 e^{-x}$.

The general solution of the original equation is $y = y_c + y_p = C_1 + C_2 e^{-x} + \cos(2x) - 2\sin(2x)$.

Example. Consider the third-order ordinary differential equation $y''' + y' = 3\sin(x)$. Let's start by solving the complementary homogeneous equation $y''' + y' = 0$:

- Characteristic polynomial: $r^3 + r = r(r^2 + 1) = r(r+i)(r-i)$.
- Roots: $r_1 = 0$, $r_2 = i$, and $r_3 = -i$.
- General solution: $y_c = C_1 + C_2\sin(x) + C_3\cos(x)$.

An attempt to find a particular solution in the form $\mathfrak{y}_p = A\cos(x) + B\sin(x)$ would be unsuccessful, as these functions are solutions of the complementary homogeneous equation and, therefore, $\mathfrak{y}_p''' + \mathfrak{y}_p' = 0$. To address this, we seek a particular solution in the form $y_p = Ax\cos(x) + Bx\sin(x)$. Computing derivatives

$$\begin{aligned} y_p' &= A\cos(x) - Ax\sin(x) + B\sin(x) + Bx\cos(x) \\ y_p'' &= -2A\sin(x) - Ax\cos(x) + 2B\cos(x) - Bx\sin(x) \\ y_p''' &= -3A\cos(x) + Ax\sin(x) - 3B\sin(x) - Bx\cos(x) \end{aligned}$$

and substituting them into the differential equation

$$\begin{aligned} &y_p''' + y_p' = 3\sin(x) \Leftrightarrow \\ &-3A\cos(x) + Ax\sin(x) - 3B\sin(x) - Bx\cos(x) + A\cos(x) - Ax\sin(x) + B\sin(x) + Bx\cos(x) = 3\sin(x) \Leftrightarrow \\ &-2A\cos(x) + (-2B - 3)\sin(x) = 0, \end{aligned}$$

we obtain the following system of equations:

$$\begin{cases} -2A = 0 \\ -2B - 3 = 0. \end{cases}$$

Solving this system, we find $A = 0$ and $B = -1.5$, yielding $y_p = -1.5x\sin(x)$. The general solution is then $y = y_c + y_p = C_1 + C_2\sin(x) + C_3\cos(x) - 1.5x\sin(x)$.

The Principle of Superposition

Consider the scenario where y_{p_1} is a particular solution of the nonhomogeneous differential equation $\mathcal{D}(y) = F_1$, and y_{p_2} is a particular solution of $\mathcal{D}(y) = F_2$. The linearity of the differential operator $\mathcal{D}$ implies that $\mathcal{D}(y_{p_1} + y_{p_2}) = \mathcal{D}(y_{p_1}) + \mathcal{D}(y_{p_2}) = F_1 + F_2$. This observation generalizes to any finite number of functions F_i.

This property allows us to decompose a nonhomogeneous equation into simpler components, find particular solutions for each component, and then combine these solutions to obtain a particular solution for the original problem.

Example. Consider the nonhomogeneous differential equation $y''' - 9y' = 30e^{2x} + 7x$. We begin by solving the complementary homogeneous equation $y''' - 9y' = 0$:

- Characteristic polynomial: $r^3 - 9r = r(r^2 - 9) = r(r-3)(r+3)$.
- Roots: $r_1 = 0$, $r_2 = 3$, and $r_3 = -3$.
- General solution: $y_c = C_1 + C_2e^{3x} + C_3e^{-3x}$.

Now, we set $F_1(x) = 30e^{2x}$ and $F_2(x) = 7x$ with $F(x) = F_1(x) + F_2(x)$ and find particular solutions of the equations $y''' - 9y' = 30e^{2x}$ and $y''' - 9y' = 7x$.

In the first case, we look for a solution in the form $y_{p_1} = Ae^{2x}$. Plugging this into the first equation gives $8Ae^{2x} - 18Ae^{2x} = 30e^{2x} \Leftrightarrow -10Ae^{2x} = 30e^{2x} \Leftrightarrow -10A = 30$, hence, $A = -3$ and $y_{p_1} = -3e^{2x}$.

Similarly, plugging $y_{p_2} = Ax^2 + Bx + C$ into the second equation produces $0 - 9(2Ax + B) = 7x \Leftrightarrow (7 + 18A)x + 9B = 0$. As the latter equality must hold true for any x, we get $A = -\frac{7}{18}$ and $B = 0$. Hence, $y_{p_2} = -\frac{7}{18}x^2$. Parameter C can be chosen arbitrarily, so we pick $C = 0$.

Finally, the solution of the initial equation is $y = y_c + y_{p_1} + y_{p_2} = C_1 + C_2e^{3x} + C_3e^{-3x} - 3e^{2x} - \frac{7}{18}x^2$.

We can systematically approach similar problems using the following step-by-step procedure.

Step 1. Solve the complementary homogeneous equation $\mathcal{D}(y) = 0$ to find the complementary solution y_c.

Step 2. Express the nonhomogeneous term $F(x)$ as a superposition (sum) of simple functions: $F(x) = F_1(x) + F_2(x) + \ldots F_k(x)$.

Step 3. Guess a particular solution y_{p_i} for each $F_i(x)$ using the method of undetermined coefficients by matching the initial form of y_{p_i} to the types of functions present in $F_i(x)$.

Step 4. Substitute each y_{p_i} into the original nonhomogeneous equation and solve for the undetermined coefficients. This may involve setting up a system of equations for each $F_i(x)$.

Step 5. Combine the complementary solution y_c with the sum of particular solutions $\sum_{i=1}^{k} y_{p_i}$ to obtain the general solution of the nonhomogeneous equation: $y = y_c + \sum_{i=1}^{k} y_{p_i}$

Lecture 4

The Method of Variation of Parameters

The method of variation of parameters offers a more general approach for determining a particular solution of the nonhomogeneous n^{th} order linear differential equation

$$\mathcal{D}(y) = y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = F(x).$$

Compared to the method of undetermined coefficients discussed in the previous lecture, the variation of parameters method is more versatile. To utilize this method, one must first solve the corresponding homogeneous equation $\mathcal{D}(y) = 0$, which may be challenging in general unless the coefficients are constants. Neither method is designed to handle this step. However, once the homogeneous equation is solved, the variation of parameters method proves more general than undetermined coefficients. It provides an expression for the particular solution for any continuous force function $F(x)$, whereas the method of undetermined coefficients is practically restricted to a limited class of force functions $F(x)$.

Example. Consider the differential equation $y'' + y = \tan(x)$ on the open interval $\left(0, \frac{\pi}{2}\right)$.

To find the complementary solution, we solve the characteristic equation $r^2 + 1 = 0$, giving $r_1 = i$ and $r_2 = -i$. Therefore, $y_c = C_1 \cos(x) + C_2 \sin(x)$.

Now, let's search for a solution in the form $y = u_1 y_1 + u_2 y_2 = u_1 \cos(x) + u_2 \sin(x)$, where u_1 and u_2 are unknown functions of x. We compute $y' = u_1' \cos(x) - u_1 \sin(x) + u_2' \sin(x) + u_2 \cos(x)$ and impose an additional condition $u_1' \cos(x) + u_2' \sin(x) = 0$. Now $y' = -u_1 \sin(x) + u_2 \cos(x)$ gives $y'' = -u_1' \sin(x) - u_1 \cos(x) + u_2' \cos(x) - u_2 \sin(x)$. The original equation becomes

$$-u_1' \sin(x) - u_1 \cos(x) + u_2' \cos(x) - u_2 \sin(x) + u_1 \cos(x) + u_2 \sin(x) = \tan(x),$$

which simplifies to

$$-u_1' \sin(x) + u_2' \cos(x) = \tan(x).$$

At this point, it remains to find two functions, u_1 and u_2, satisfying the system of equations

$$\begin{cases} -u_1' \sin(x) + u_2' \cos(x) = \tan(x) \\ u_1' \cos(x) + u_2' \sin(x) = 0. \end{cases}$$

From the second equation, we express $u_2' = -\frac{\cos(x)}{\sin(x)} u_1'$, and the first equation becomes

$$-\left(\sin(x) + \frac{\cos^2(x)}{\sin(x)}\right) u_1' = \tan(x).$$

Using a chain of trigonometric identities, we obtain $u_1' = -\frac{\tan(x)}{\sin(x) + \frac{\cos^2(x)}{\sin(x)}} = -\frac{\sin(x)\tan(x)}{\sin^2(x) + \cos^2(x)} = -\sin(x)\tan(x) = -\frac{\sin^2(x)}{\cos(x)} = -\frac{1-\cos^2(x)}{\cos(x)} = -\sec(x) + \cos(x)$. This allows us to find

$$u_1 = \int (-\sec(x) + \cos(x))\, dx = -\ell n |\tan(x) + \sec(x)| + \sin(x) + C_1.$$

Similarly, $u_2' = -\frac{\cos(x)}{\sin(x)} u_1' = \frac{\cos(x)}{\sin(x)} \cdot \frac{\sin^2(x)}{\cos(x)} = \sin(x)$, with

$$u_2 = \int \sin(x)\, dx = -\cos(x) + C_2.$$

Finally, collecting everything together, we obtain the general solution to the initial equation:

$$\begin{aligned} y &= u_1 \cos(x) + u_2 \sin(x) = (-\ell n|\tan(x) + \sec(x)| + \sin(x) + C_1)\cos(x) + (-\cos(x) + C_2)\sin(x) = \\ &= C_1 \cos(x) + C_2 \sin(x) - \cos(x)\ell n|\tan(x) + \sec(x)| + \sin(x)\cos(x) - \cos(x)\sin(x) = \\ &= C_1 \cos(x) + C_2 \sin(x) - \cos(x)\ell n|\tan(x) + \sec(x)|. \end{aligned}$$

Next, we generalize the method to arbitrary linear nonhomogeneous equations of order two:

$$y'' + p(x)y' + q(x)y = F(x).$$

This process is straightforward. After finding the general solution $y_c = C_1y_1 + C_2y_2$ to the complementary equation $y'' + p(x)y' + q(x)y = 0$, we look for a particular solution in the form $y = u_1y_1 + u_2y_2$, where u_1 and u_2 are unknown functions of x. We compute $y' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$ and impose an additional condition $u_1'y_1 + u_2'y_2 = 0$. After straightforward algebraic manipulations, the initial equation turns into $u_1'y_1' + u_2'y_2' = F(x)$, and we are left with finding u_1 and u_2 from the system of equations

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0, \\ u_1'y_1' + u_2'y_2' = F(x). \end{cases}$$

We can express $u_2' = -\frac{y_1}{y_2}u_1'$ and then find $u_1' = \frac{F(x)}{y_1' - \frac{y_1y_2'}{y_2}} = \frac{y_2F(x)}{y_2y_1' - y_1y_2'} = \frac{y_2F(x)}{W(y_1,y_2)}$ from the first equation. Hence,

$$u_1 = \int \frac{y_2F(x)}{W(y_1,y_2)}\,dx \quad \text{and} \quad u_2 = \int \frac{y_1F(x)}{W(y_1,y_2)}\,dx,$$

where $W(y_1,y_2) = y_2y_1' - y_1y_2'$ is the Wronskian of y_1 and y_2. Finally,

$$y = y_1 \int \frac{y_2F(x)}{W(y_1,y_2)}\,dx + y_2 \int \frac{y_1F(x)}{W(y_1,y_2)}\,dx.$$

If the integrals above can be expressed in terms of elementary functions, we obtain specific functions; otherwise, the solution remains in integral form.

General n

We begin by exploring the impact of varying parameters on higher-order differential equations. The process involves finding the general solution $y_c = C_1y_1 + C_2y_2 + \ldots + C_ny_n$ for the complementary equation $D(y) = 0$. Subsequently, we seek a particular solution in the form $y = u_1y_1 + u_2y_2 + \ldots + u_ny_n$, with the functions $u_1, u_2, \ldots, u_n$ to be determined.

To derive the solution, we compute the first derivative:

$$y' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2' + \ldots + u_n'y_n + u_ny_n'$$

We then impose the condition

$$u_1'y_1 + u_2'y_2 + \ldots + u_n'y_n = 0.$$

Simplifying, we obtain

$$y' = u_1y_1' + u_2y_2' + \ldots + u_ny_n'.$$

The process is repeated for higher derivatives, imposing conditions until the $(n-1)$-th derivative, resulting in

$$y^{(n-1)} = u_1y_1'' + u_2y_2'' + \ldots + u_ny_n''$$

This establishes a system of $(n-1)$ additional conditions on $u_1', u_2', \ldots, u_n'$:

$$u_1'y_1^{(i)} + u_2'y_2^{(i)} + \ldots + u_n'y_n^{(i)} \quad \text{for } i \in \{0, 1, 2, \ldots, n-2\}.$$

By plugging these expressions for $y', y'', \ldots, y^{(n-1)}$ into the nonhomogeneous equation $\mathcal{D}(y) = F(x)$, we transform it into

$$y_1^{(n-1)}u_1 + y_2^{(n-1)}u_2 + \ldots + y_n^{(n-1)}u_n = F(x).$$

The unknown functions $u'_1, u'_2, \ldots, u'_n$ (and subsequently $u_1, u_2, \ldots, u_n$) can then be determined from the system of n equations:

$$\begin{cases} y_1 u'_1 + y_2 u'_2 + \ldots + y_n u'_n = 0 \\ y'_1 u'_1 + y'_2 u'_2 + \ldots + y'_n u'_n = 0 \\ y''_1 u'_1 + y''_2 u'_2 + \ldots + y''_n u'_n = 0 \\ \vdots \\ y_1^{(n-2)} u'_1 + y_2^{(n-2)} u'_2 + \ldots + y_n^{(n-2)} u'_n = 0 \\ y_1^{(n-1)} u'_1 + y_2^{(n-1)} u'_2 + \ldots + y_n^{(n-1)} u'_n = F(x) \end{cases}$$

Notice that the determinant of the matrix of coefficients is exactly the Wronskian of the functions $y_1, y_2, \ldots, y_n$:

$$W(y_1, y_2, \ldots, y_n)(x) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y'_1 & y'_2 & \cdots & y'_n \\ y''_1 & y''_2 & \cdots & y''_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

As the functions $y_1, y_2, \ldots, y_n$ form a fundamental set of solutions, we must have that the Wronskian is not identically zero. Hence, the matrix of coefficients is invertible on some interval (where $W(y_1, y_2, \ldots, y_n)(x)$ has no zeros). It follows that on this interval $(u'_1, u'_2, \ldots, u'_n) = W^{-1}(0, 0, \ldots, 0, F(x))$. Alternatively, using Cramer's rule, we can express

$$u'_i(x) = \frac{\widetilde{W}_i(x)}{W(x)},$$

where the matrix $\widetilde{W}_i(x)$ is the determinant of the matrix obtained from W by substituting the m^{th} column with $(0, 0, \ldots, 0, F(x))$. Using $(0, 0, \ldots, 0, F(x)) = F(x)(0, 0, \ldots, 0, 1)$ and setting $W_i(x)$ to be the determinant of the matrix obtained from $W(x)$ by substituting the m^{th} column with $(0, 0, \ldots, 0, 1)$, we can write

$$u'_i(x) = \frac{F(x) W_i(x)}{W(x)}.$$

The general solution to the initial nonhomogeneous equation $\mathcal{D}(y) = F(x)$ can be expressed as

$$y = \sum_{i=1}^{n} y_i \int \frac{F(x) W_i(x)}{W(x)}\, dx.$$

Question. Why do we choose the conditions $u'_1 y_1^{(i)} + u'_2 y_2^{(i)} + \ldots + u'_n y_n^{(i)}$ for $i \in \{0, 1, 2, \ldots, n-2\}$? How do these restrictions lead to the system of equations derived above?

To understand the rationale behind these conditions, consider the product rule when taking a derivative of a summand, $(y_i^{(j)} u'_i)' = y_i^{(j+1)} u'_i + y_i^{(j)} u''_i$. The imposed conditions essentially serve a dual purpose. Firstly, they prevent the proliferation of new unknowns, specifically higher derivatives of u_i's. By setting the cumulative sum of these terms to zero, we effectively restrict the spread of additional unknowns.

Simultaneously, these conditions allow only higher derivatives of y_i to participate. This choice is made strategically to naturally construct a system of coefficients resembling the structure of the matrix used for computing the Wronskian. In essence, the conditions are designed to streamline the system of equations, leading to a more manageable and interpretable solution.

It would be great to stop here, but I owe you at least one concrete example 🙂

Example. We are going to solve the third-order linear differential equation

$$y''' + y' = \sec(x),$$

defined on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Step 1: Solve the complementary equation $y''' + y' = 0$.

- Characteristic polynomial: $r^3 + r = r(r-i)(r+i)$.
- Roots: $r_1 = 0$, $r_2 = i$, and $r_3 = -i$.
- General solution: $y_c = C_1 + C_2\cos(x) + C_3\sin(x)$, with fundamental set $y_1 = 1$, $y_2 = \cos(x)$, and $y_3 = \sin(x)$.

Step 2: Compute the Wronskian $W(y_1, y_2, y_3)$.

$$W(y_1,y_2,y_3)(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \sin^2(x) + \cos^2(x) = 1.$$

Step 3: Compute the determinants of auxiliary matrices.

$$W_1(y_1,y_2,y_3)(x) = \det\begin{pmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ 1 & y_2'' & y_3'' \end{pmatrix} = \det\begin{pmatrix} 0 & \cos(x) & \sin(x) \\ 0 & -\sin(x) & \cos(x) \\ 1 & -\cos(x) & -\sin(x) \end{pmatrix} = \sin^2(x) + \cos^2(x) = 1,$$

$$W_2(y_1,y_2,y_3)(x) = \det\begin{pmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & 1 & y_3'' \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & \sin(x) \\ 0 & 0 & \cos(x) \\ 0 & 1 & -\sin(x) \end{pmatrix} = -\cos(x),$$

$$W_3(y_1,y_2,y_3)(x) = \det\begin{pmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & \cos(x) & 0 \\ 0 & -\sin(x) & 0 \\ 0 & -\cos(x) & 1 \end{pmatrix} = -\sin(x).$$

Step 4: Find the functions $u_1(x)$, $u_2(x)$, and $u_3(x)$.

$$u_1(x) = \int \frac{F(x)W_1(x)}{W(x)} = \int \sec(x)\,dx = \ell n|\sec(x) + \tan(x)| + C_1,$$

$$u_2(x) = \int \frac{F(x)W_2(x)}{W(x)} = \int -\,dx = -x + C_2,$$

$$u_3(x) = \int \frac{F(x)W_3(x)}{W(x)} = \int -\tan(x)\,dx = \ell n|\cos(x)| + C_3.$$

Step 5: Obtain the solution y.

$$\begin{aligned} y = y_1u_1 + y_2u_2 + y_3u_3 &= 1\cdot(\ell n|\sec(x)+\tan(x)| + C_1) + \cos(x)(-x + C_2) + \sin(x)(\ell n|\cos(x)| + C_3) \\ &= C_1 + C_2\cos(x) + C_3\sin(x) + \ell n|\sec(x)+\tan(x)| - x\cos(x) + \sin(x)\ell n|\cos(x)|. \end{aligned}$$

Lecture 5

The Power of Series

In the upcoming lectures, we will explore the application of power series in the construction of fundamental sets of solutions for second-order linear differential equations, where the coefficients in these equations are functions of the independent variable.

To kick things off, let's take a moment to provide a concise overview of the essential results and concepts related to power series that will serve as the foundation for our discussions.

Definition. A **power series about** $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n + \ldots,$$

where the **center** a and the coefficients $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

How to Test a Power Series for Convergence

For a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, if

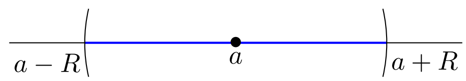
$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L,$$

then the series converges absolutely if $0 < L < 1$, diverges if $L > 1$, and the test is inconclusive if $L = 1$.

Radius of Convergence. The radius of convergence is given by

$$R = \frac{1}{\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|}.$$

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$, then $R = \infty$ (the power series converges everywhere). If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $R = 0$ (the power series converges only at the center of the series).



Example. 1. Taking all coefficients to be 1 in the power series centered at $x = 0$ gives the geometric power series:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$$

This is the geometric series with the first term 1 and ratio x. To compute the radius of convergence, we will use the ratio test:

$$L = \lim_{n\to\infty}\left|\frac{c_{n+1}x^{n+1}}{c_n x^n}\right| = |x| \leq 1.$$

The series converges when $|x| < 1$, and diverges when $|x| > 1$. Therefore, the radius of convergence is $R = 1$.

Now, for the boundary points:

for $x = 1$, the series becomes $1 + 1 + 1 + \ldots$, which diverges;

for $x = -1$, the series becomes $1 - 1 + 1 - 1 + \ldots$, which does not converge either.

Therefore, the series diverges for $x = 1$ and $x = -1$.

2. Consider the power series

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24} + \ldots + \ldots.$$

The radius of convergence can be found by applying the ratio test:

$$\lim_{n\to\infty}\left|\frac{x^{2(n+1)}/(n+1)!}{x^{2n}/n!}\right| = \lim_{n\to\infty}\frac{x^2}{n+1} = 0.$$

The limit is always 0, less than 1, indicating convergence for all x. Therefore, the radius of convergence R is infinite.

The series converges for all values of x, and there are no boundary points to consider.

Optional: Riemann Rearrangement Theorem

Note: This section is optional.

Since the ancient era when Zeno set Achilles on the seemingly endless pursuit of a tortoise, infinite series have captivated our imagination and curiosity. These mathematical constructs have a unique ability to challenge and sometimes contradict our intuitive understanding of numbers and the natural world. Zeno's paradoxes, originating from this pursuit, continue to intrigue us, despite the fact that the fallacies in his arguments have long been identified.

In the late seventeenth and eighteenth centuries, mathematicians often found themselves puzzled by the outcomes generated while working with infinite series. It wasn't until the nineteenth century that the realization dawned—divergent series were frequently the culprits behind these perplexities.

Neils Hendrik Abel, a prominent mathematician of the time, expressed his frustration in a letter to a friend in 1826, stating, "Divergent series are the invention of the devil. By using them, one may draw any conclusion he pleases, and that is why these series have produced so many fallacies and so many paradoxes."

Example. Consider the alternating harmonic series

$$\sum_{n=1}^{\infty}(-1)^n \cdot \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

As $\lim\limits_{n\to\infty} \frac{1}{n} = 0$, the alternating series test assures the convergence of the series. Let $\mathcal{S} := \sum\limits_{n=1}^{\infty}(-1)^n \cdot \frac{1}{n}$. Consider the following observation:

$$2\mathcal{S} = 2 - 1 + \frac{2}{3} - \frac{2}{4} + \frac{2}{5} - \frac{2}{6} + \frac{2}{7} - \frac{2}{8} + \frac{2}{9} - \frac{2}{10} + \dots = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots = \mathcal{S}$$

Here, we used the fact that $\frac{2}{k} - \frac{2}{2k} = \frac{1}{k}$ for odd k and $\frac{2}{2s} = \frac{1}{s}$ for even $k = 2s$.

The equality $2\mathcal{S} = \mathcal{S}$ implies $2 = 1$ unless $\mathcal{S} = 0$. Furthermore, observe that $\sum\limits_{n=1}^{\infty}(-1)^n \cdot \frac{1}{n}$ is the Taylor series for $\ell n(1 + x)$ evaluated at $x = 1$, and hence, $\mathcal{S} = \ell n(2) \neq 0 \dots$.

Peter Lejeune-Dirichlet made this remarkable discovery in 1827, while investigating conditions for the convergence of Fourier series. He was the first to observe that it is possible to rearrange the terms of certain series, now known as conditionally convergent series, in such a way that the sum changes. The intriguing aspect was that Dirichlet couldn't provide an explanation for why this result was possible. (In 1837, he did prove that rearranging the terms of an absolutely convergent series does not alter its sum.) This discovery opened the path for Dirichlet to address the convergence of Fourier series, and by 1829, he successfully solved one of the prominent problems of that era: providing sufficient conditions for a real-valued, periodic function f to be equal to the sum of its Fourier series.

In 1852, Bernhard Riemann took up the mantle, working on extending Dirichlet's results on the convergence of Fourier series. Seeking Dirichlet's advice, Riemann showed him a draft of his work. During this collaboration, Dirichlet reminisced about his own work on the problem, sharing the insight that rearranging the terms of a conditionally convergent series could alter its sum. Riemann suspected that divergent series played a role in this phenomenon. He went on to develop a remarkable explanation, now known as Riemann's rearrangement theorem, which he incorporated into his paper on Fourier series. Although completed in 1853, the paper was not published until after Riemann's death in 1866 under the title "On the Representation of a Function by a Trigonometric Series."

Riemann's own description of the theorem and its proof, translated from German, reads:

... infinite series fall into two distinct classes, depending on whether or not they remain convergent when all the terms are made positive. In the first class the terms can be arbitrarily rearranged; in the second, on the other hand, the ***value is dependent on the ordering of the terms****. Indeed, if we denote the positive terms of a series in the second class by* $a_1, a_2, a_3, \dots$ *and the negative terms by* $-b_1, -b_2, -b_3, \dots$, *then it is clear that* $\sum\limits_{i=1}^{\infty} a_i$ *as well as* $\sum\limits_{i=1}^{\infty} b_i$ *must be infinite. For if they were both finite, the series would still*

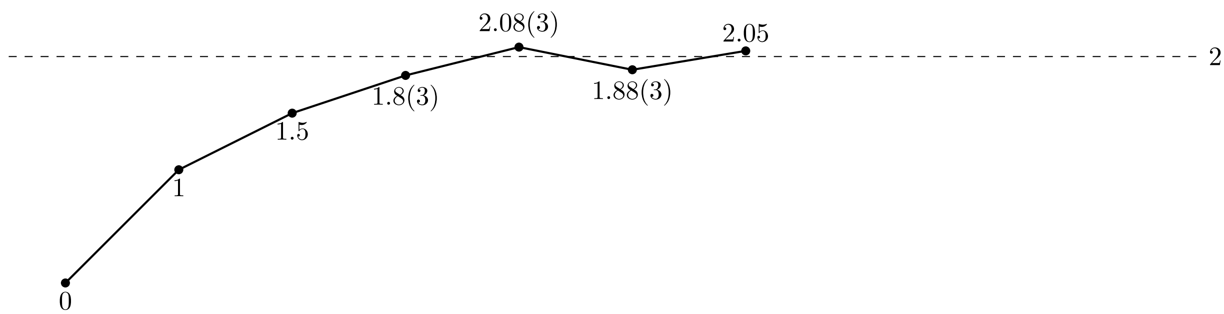
be convergent after making all the signs the same. If only one were infinite, then the series would diverge. Clearly now an arbitrarily given value C can be obtained by a suitable reordering of the terms. We take alternately the positive terms of the series until the sum is greater than C, and then the negative terms until the sum is less than C. The deviation from C never amounts to more than the size of the term at the last place the signs were switched. Now, since the numbers a as well as the numbers b become infinitely small with increasing index, so also are the deviations from C. If we proceed sufficiently far in the series, the deviation becomes arbitrarily small, that is, the series converges to C.

Example. Suppose we wish to arrange the signs in the series $\sum_{n=1}^{\infty} \frac{a_n}{n}$, where each a_n is either $+1$ or -1, in such a way that the series converges to 2. We start by setting $a_1 = 1$, leading to the partial sum $S_1 = 1$. Since $S_1 < 2$, we decide to continue with $a_2 = 1$, resulting in $S_2 = 1 + \frac{1}{2} = 1.5$.

Seeing that $S_2 < 2$, we choose $a_3 = 1$ as well, giving $S_3 = 1.5 + \frac{1}{3} = 1.8(3)$. Continuing this process, we find that $S_4 = 1.8(3) + \frac{1}{4} = 2.08(3)$. Now, $S_4 > 2$, indicating that we need to adjust the signs.

To bring the sum closer to 2, we choose $a_5 = -1$, resulting in $S_5 = 2.08(3) - 0.2 = 1.88(3)$. This process continues, with alternating signs being selected whenever the partial sum 'crosses' 2.

The iterative nature of this procedure allows us to fine-tune the series to converge to the desired value, 2, by strategically adjusting the signs of each term in the sequence.



Lecture 6
Taylor Series, Analytic Functions and Solutions Near an Ordinary Point

If $\sum_{n=0}^{\infty} c_n(x-a)^n$ has a radius of convergence $R > 0$, it defines a function $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ on the interval $a - R < x < a + R$. The function $f(x)$ has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1},$$
$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2}, \text{etc.}$$

The value of c_n is given by $c_n = \frac{f^{(n)}(a)}{n!}$. The series is called the **Taylor series** for the function f about $x = a$.

Analytic Functions

A function f that has a Taylor series expansion about $x = a$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

with a radius of convergence $R > 0$, is said to be **analytic** at $x = a$. All familiar functions of calculus are analytic except perhaps at certain easily recognized points.

For instance, e^x, $\sin(x)$, and $\cos(x)$ are analytic everywhere. The function $1/(x-5)$ is analytic everywhere except at $x = 5$, while $\tan(x)$ is analytic everywhere except at odd multiples of $\pi/2$.

If f and g are analytic at a, then so are $f \pm g$, $f \cdot g$, and f/g (provided that $g(a) \neq 0$).

Example. 1. Determine the Taylor series about the point $x_0 = -1$ for the function $g(x) = 2x - x^3$ and find the radius of convergence for the corresponding series.

$$\begin{array}{llll}
g(x) = 2x - x^3 & g(-1) = -1 & & c_0 = \dfrac{g(-1)}{0!} = -1 \\
g'(x) = 2 - 3x^2 & g'(-1) = -1 & & c_1 = \dfrac{g'(-1)}{1!} = -1 \\
g''(x) = -6x & g''(-1) = 6 & & c_2 = \dfrac{g''(-1)}{2!} = 3 \\
g'''(x) = -6 & g'''(-1) = -6 & & c_3 = \dfrac{g'''(-1)}{3!} = -1 \\
g^{(n)}(x) = 0 & g^{(n)}(-1) = 0 & \text{for } n > 3 & c_n = 0
\end{array}$$

This gives the Taylor series expansion

$$g(x) = -1 - (x+1) + 3(x+1)^2 - (x+1)^3.$$

Since the expression is finite, it converges for every value of x, i.e., $R = \infty$.

2. Consider the function $f(x) = \frac{1}{1-x}$, and let's determine its Taylor series about the point $x_0 = 2$. We will also find the radius and interval of convergence for the series.

$$\begin{array}{ll}
f(x) = \dfrac{1}{1-x} & f(2) = \dfrac{1}{1-2} = -1, \\
f'(x) = \dfrac{1}{(1-x)^2} & f'(2) = \dfrac{1}{(1-2)^2} = 1, \\
f''(x) = \dfrac{2}{(1-x)^3} & f''(2) = \dfrac{2}{(1-2)^3} = -2, \\
f'''(x) = \dfrac{6}{(1-x)^4} & f'''(2) = \dfrac{6}{(1-2)^4} = 6, \\
\dots & \\
f^{(n)}(x) = \dfrac{n!}{(1-x)^{n+1}} & f^{(n)}(2) = \dfrac{(-1)^{n+1} n!}{n+1},
\end{array}$$

giving $c_n = \frac{f^{(n)}(2)}{n!} = (-1)^{n+1}$ and the Taylor series expansion

$$f(x) = \sum_{n=0}^{\infty} (-1)^{n+1} (x-2)^n.$$

To determine the radius of convergence, we use the ratio test:

$$R = \frac{1}{\lim\limits_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|} = \frac{1}{1} = 1.$$

This guarantees convergence on the open interval $(1, 3)$, consisting of points within 1 from $x_0 = 2$. It remains to check convergence for the boundary points $x = 1$ and $x = 3$. In the former case, we get $\sum\limits_{n=0}^{\infty} (-1)^{n+1}(1-2)^n = \sum\limits_{n=0}^{\infty} (-1)^{2n+1} = \sum\limits_{n=0}^{\infty} (-1)$, and in the latter, $\sum\limits_{n=0}^{\infty} (-1)^{n+1}(3-2)^n = \sum\limits_{n=0}^{\infty} (-1)^{n+1}$, both of which diverge. Hence, the interval of convergence is $(1, 3)$.

Shift of Index of Summation

In an infinite series, the index of summation is a dummy parameter, similar to the integration variable in a definite integral. Consequently, the specific letter used for the index is arbitrary. For example,

$$\sum_{n=0}^{\infty} \frac{n^3 x^n}{n!} = \sum_{j=0}^{\infty} \frac{j^3 x^j}{j!}.$$

Just as changes of variables are made in definite integrals, we find it convenient to make changes of summation indices when calculating series solutions of differential equations. Let's illustrate the process of shifting the summation index by an example.

Example. 1. Express the series $\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n$ as a sum with generic term x^n. Introducing the dummy variable $m = n-1$ instead of n, rewrite $\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m$. This leads to the expression $\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (n a_n + a_{n+1}) x^n$ as desired.

2. Check the equality

$$\sum_{k=0}^{\infty} c_{k+1} x^k + \sum_{k=0}^{\infty} c_k x^{k+1} = c_1 + \sum_{k=1}^{\infty} (c_{k+1} + c_{k-1}) x^k.$$

To verify this equality, let's focus on the first term:

$$\sum_{k=0}^{\infty} c_{k+1} x^k = a_1 x^0 + \sum_{k=1}^{\infty} c_{k+1} x^k = a_1 + \sum_{k=1}^{\infty} c_{k+1} x^k \quad \text{(shifting index)}.$$

Now, we add the second term and check:

$$\sum_{k=0}^{\infty} c_{k+1} x^k + \sum_{k=0}^{\infty} c_k x^{k+1} = c_1 + \sum_{k=1}^{\infty} c_{k+1} x^k + \sum_{k=1}^{\infty} c_k x^{k+1} = c_1 + \sum_{k=1}^{\infty} (c_{k+1} + c_{k-1}) x^k \quad \text{(combine terms)}.$$

Hence, the equality holds.

Series Solutions Near an (ordinary) Point

Let's consider methods for solving second-order linear equations when the coefficients are functions of the independent variable, denoted as x in this chapter. We will focus on the homogeneous equation:

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

This treatment will suffice, as the approach for the nonhomogeneous counterpart is similar. Many mathematical physics problems yield such equations with polynomial coefficients. For instance, Bessel equation, encountered in heat conduction:

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0$$

where ν is a constant.

To simplify computations and accommodate various scenarios, we focus on polynomial functions P, Q, and R. However, note that the solution method remains largely unchanged when P, Q, and R are general analytic functions.

For now, assume that P, Q, and R are polynomials without common factors. The solution in an interval containing a point x_0 depends on P's behavior in that interval. If $P(x_0) \neq 0$, x_0 is called an *ordinary* point; otherwise, it's *singular*. Let's concentrate on solutions in the neighborhood of ordinary points first.

Since P is continuous, there exists an interval around x_0 where $P(x)$ is never zero. In this interval, we can divide the initial equation by $P(x)$ to obtain:

$$y'' + p(x)y' + q(x)y = 0$$

where $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$ are continuous rational functions (ratios of polynomials). According to the existence and uniqueness theorem, in this interval, there exists a unique solution that also satisfies the boundary conditions $y(x_0) = y_0$ and $y'(x_0) = y'_0$ for arbitrary values of y_0 and y'_0.

We will look for solutions in the form

$$y = c_0 + c_1(x - x_0) + \ldots + c_n(x - x_0)^n + \ldots = \sum_{n=0}^{\infty} c_n(x - x_0)^n$$

and make the assumption that this series converges within the interval $|x - x_0| < \rho$ for some $\rho > 0$.

At first glance, adopting a power series as the form of a solution might seem less intuitive. However, it proves to be a remarkably convenient and practical choice. Power series, within their intervals of convergence, exhibit behavior akin to polynomials and can be manipulated in a straightforward way. Even when solutions can be expressed in terms of elementary functions like exponentials or trigonometric functions, employing a power series or an equivalent form becomes crucial for numerical evaluations or graphical plotting.

A pragmatic approach for determining the coefficients c_n involves substituting the series and its derivatives for y, y', and y'' into the original equation followed by recursively finding the coefficients.

Example. Consider the differential equation $y'' - xy' - y = 0$. To find the power series solution centered at $x_0 = 1$, let's express the solution in the form:

$$y = \sum_{n=0}^{\infty} c_n(x - 1)^n$$

In addition, we compute

$$y' = c_1 + 2c_2(x - 1) + 3c_3(x - 1)^2 + \ldots + (n + 1)c_{n+1}(x - 1)^n + \ldots = \sum_{n=0}^{\infty}(n + 1)c_{n+1}(x - 1)^n$$

$$y'' = 2c_2 + 6c_3(x - 1) + \ldots + (n + 2)(n + 1)c_{n+2}(x - 1)^n + \ldots = \sum_{n=0}^{\infty}(n + 2)(n + 1)c_{n+2}(x - 1)^n.$$

Next we perform a sequence of algebraic manipulations to express the latter equation in a more convenient form. By writing $x = (x - 1 + 1)$, we can transform the middle term as follows:

$$x\sum_{n=0}^{\infty}(n + 1)c_{n+1}(x - 1)^n = ((x - 1) + 1)\sum_{n=0}^{\infty}(n + 1)c_{n+1}(x - 1)^n = \sum_{n=0}^{\infty}(n + 1)c_{n+1}(x - 1)^{n+1} +$$
$$\sum_{n=0}^{\infty}(n + 1)c_{n+1}(x - 1)^n = \sum_{n=1}^{\infty} nc_n(x - 1)^n + \sum_{n=0}^{\infty}(n + 1)c_{n+1}(x - 1)^n = \sum_{n=0}^{\infty}(nc_n + (n + 1)c_{n+1})(x - 1)^n.$$

Substituting these into the original equation, we get:

$$\sum_{n=0}^{\infty}(n + 2)(n + 1)c_{n+2}(x - 1)^n - x\sum_{n=0}^{\infty}(n + 1)c_{n+1}(x - 1)^n - \sum_{n=0}^{\infty} c_n(x - 1)^n = 0 \Leftrightarrow$$
$$\sum_{n=0}^{\infty}(n + 2)(n + 1)c_{n+2}(x - 1)^n - \sum_{n=0}^{\infty}(nc_n + (n + 1)c_{n+1})(x - 1)^n - \sum_{n=0}^{\infty} c_n(x - 1)^n = 0 \Leftrightarrow$$
$$\sum_{n=0}^{\infty}((n + 2)(n + 1)c_{n+2} - (n + 1)c_{n+1} - (n + 1)c_n)(x - 1)^n = 0,$$

For a power series to be identically zero on an interval, it is essential that all of its coefficients must vanish. This leads to a system of linear equations:

$$\begin{cases} 2c_2 - c_1 - c_0 = 0 \\ 3c_3 - c_2 - c_1 = 0 \\ \dots \\ (n+2)c_{n+2} - c_{n+1} - c_n = 0 \\ \dots \end{cases}$$

The coefficients c_n are uniquely determined once the values of c_0 and c_1 are set:

$$\begin{aligned} c_2 &= \frac{c_0 + c_1}{2} = \frac{c_0}{2} + \frac{c_1}{2} \\ c_3 &= \frac{c_1 + c_2}{3} = \frac{c_1 + \frac{c_0}{2} + \frac{c_1}{2}}{3} = \frac{c_0}{6} + \frac{c_1}{2} \\ &\dots \end{aligned}$$

Notice that for any analytic solution $y = \sum_{n=0}^{\infty} c_n(x-1)^n$, we have $y(1) = c_0$ and $y'(1) = c_1$. Now, consider two specific solutions y_1 with $c_0 = 1, c_1 = 0$ and y_2 with $c_0 = 0, c_1 = 1$. Evaluating the Wronskian at $x_0 = 1$, we get $W(y_1, y_2)(1) = (y_1 y_2' - y_1' y_2)(1) = 1 \cdot 1 - 0 \cdot 0 = 1$, confirming that $\{y_1, y_2\}$ form a fundamental set of solutions on a small enough interval containing $x_0 = 1$.

Lecture 7
Series Solutions Near an Ordinary Point, II

In the last lecture, we explained how to find power series solutions of second-order linear ordinary differential equations

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

with polynomial coefficients $P(x)$, $Q(x)$, and $R(x)$ near a point x_0 where $P(x_0) \neq 0$. We referred to such a point as ordinary. Today, we will expand this method to a broader class of functions, allowing $P(x)$, $Q(x)$, and $R(x)$ to be not necessarily polynomials.

Ordinary $\to$ Extraordinary

As before, we divide the initial equation by $P(x)$ to get the equation

$$y'' + p(x)y' + q(x)y = 0,$$

where $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$, and look for solutions in the form $y = \sum_{n=0}^{\infty} c_n(x - x_0)^n$.

Using that $y' = \sum_{n=0}^{\infty} (n+1)c_{n+1}(x-x_0)^n$ and $y'' = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}(x-x_0)^n$, we can derive a system of linear equations for the unknown coefficients c_n by substituting the power series and its derivatives into the initial differential equation and by taking advantage of the fact that the expression $y'' + p(x)y' + q(x)y$ along with its derivatives evaluate to zero at x_0. Let's examine the first such equation obtained from $(y'' + py' + qy)(x_0) = 0$.

$$\begin{aligned} &\left(\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}(x-x_0)^n + p(x)\sum_{n=0}^{\infty}(n+1)c_{n+1}(x-x_0)^n + q(x)\sum_{n=0}^{\infty} c_n(x-x_0)^n \right)(x_0) \\ &= 2c_2 + p(x_0)c_1 + q(x_0)c_0 = 0 \Leftrightarrow c_2 = \frac{-p(x_0)c_1 - q(x_0)c_0}{2}. \end{aligned}$$

The second equation comes from $(y''+py'+qy)'(x_0)=0 \Leftrightarrow y'''+py''+p'y'+qy'+q'y = y'''+py''+(p'+q)y'+q'y=0$ and reads

$$\left(\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}(x-x_0)^n + p(x)\sum_{n=0}^{\infty}(n+1)c_{n+1}(x-x_0)^n + q(x)\sum_{n=0}^{\infty}c_n(x-x_0)^n\right)'(x_0)$$
$$= \left(\sum_{n=0}^{\infty}(n+3)(n+2)(n+1)c_{n+3}(x-x_0)^n + p(x_0)\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}(x-x_0)^n\right.$$
$$\left. +\ (p'(x_0)+q(x_0))\sum_{n=0}^{\infty}(n+1)c_{n+1}(x-x_0)^n + q(x_0)\sum_{n=0}^{\infty}(n+1)c_{n+1}(x-x_0)^n\right)(x_0)$$
$$= 6c_3 + 2p(x_0)c_2 + (p'(x_0)+q(x_0))c_1 + q'(x_0)c_0 = 0 \Leftrightarrow c_3 = \frac{-(2p(x_0)c_2 + (p'(x_0)+q(x_0))c_1 + q'(x_0)c_0)}{6},$$

etc.

Notice that the crucial condition for generating equations on coefficients c_n is the existence of the n^{th} derivatives of the functions p and q. Therefore, to obtain the complete power series, it is essential that the derivatives of all orders of p and q at x_0 exist. This requirement holds true, in particular, for rational functions p and q with polynomials in the denominators not vanishing at x_0 that we discussed in the last class.

However, merely demanding that these functions be infinitely differentiable in the neighborhood of x_0 is necessary but insufficient. This is because it does not guarantee the convergence of the resulting series expansion for y. We need the functions p and q to be *analytic* at x_0; that is, they should have Taylor series expansions that converge to them in some interval around the point x_0:

$$p(x) = \sum_{n=0}^{\infty} p_n(x-x_0)^n \text{ and } q(x) = \sum_{n=0}^{\infty} q_n(x-x_0)^n.$$

We refine our previous definition: if the functions $p = Q/P$ and $q = R/P$ are analytic at x_0, then x_0 is said to be an *ordinary point* of the differential equation $P(x)y''+Q(x)y'+R(x)y$, otherwise, it is a *singular point*.

The following fundamental result was established by Lazarus Fuchs in 1866.

Theorem. Let x_0 be an ordinary point of the differential equation

$$P(x)y''+Q(x)y'+R(x)y=0,$$

meaning that $p=\frac{Q}{P}$ and $q=\frac{R}{P}$ are analytic at x_0. In this case, the general solution can be expressed as

$$y=\sum_{n=0}^{\infty}c_n(x-x_0)^n = c_0y_1(x)+c_1y_2(x),$$

where c_0 and c_1 are arbitrary constants, and y_1 and y_2 are two power series solutions that are analytic at x_0. These solutions y_1 and y_2 together form a fundamental set of solutions. Moreover, the radius of convergence for each of the series solutions y_1 and y_2 is at least as large as the minimum of the radii of convergence of the series for p and q.

The proof of this theorem involves technical details, and we will omit it. However, the nuances of the theorem deserve a more thorough exploration. The equality $\sum\limits_{n=0}^{\infty}c_n(x-x_0)^n = c_0y_1(x)+c_1y_2(x)$ at $x=x_0$, yielding $c_0 = c_0y_1(x_0)+c_1y_2(x_0)$. As this equality must hold for arbitrary values of c_0 and c_1, we conclude $y_1(x_0)=1$ and $y_2(x_0)=0$. Similarly, the equality $\left(\sum\limits_{n=0}^{\infty}c_n(x-x_0)^n\right)' = (c_0y_1(x)+c_1y_2(x))'$ at x_0 gives $c_1 = c_0y_1'(x_0)+c_1y_2'(x_0)$, implying $y_1'(x_0)=0$ and $y_2'(x_0)=1$. Consequently, the Wronskian

$W(y_1, y_2)(x_0) = (y_1 y_2' - y_1' y_2)(x_0) = 1 \cdot 1 - 0 \cdot 0 = 1$ does not vanish at x_0, confirming that the functions $\{y_1, y_2\}$ indeed form a fundamental set of solutions.

Another crucial point is that while the process of calculating coefficients by successive differentiation was instrumental in establishing a theoretical result, it is often impractical for computations. Instead, one should substitute the power series expression with unknown coefficients for y into the differential equation and determine the coefficients recursively, as outlined in the previous lecture.

A very important takeaway from the theorem is the existence of a series solution with a radius of convergence at least as large as the smallest of the radii of convergence of the series for p and q. Therefore, it suffices to determine those two radii.

Rational Functions: Determining Taylor Series and Radius of Convergence

Let $P(x)$ and $Q(x)$ be two polynomials, with $\deg(P) < \deg(Q)$, and $Q(x)$ having distinct roots $x_1, x_2, \ldots, x_n$. For simplicity, we assume none of the x_i's is zero. Consider the rational function $f(x) = \frac{P(x)}{Q(x)}$. Then, $f(x)$ can be expressed as $f(x) = \frac{A_1}{x_1 - x} + \frac{A_2}{x_2 - x} + \ldots + \frac{A_n}{x_n - x}$, where $A_1, A_2, \ldots, A_n$ are constants. Each term $\frac{A_i}{x_i - x}$ can be further rewritten as $\alpha_i \cdot \frac{1}{1 - \frac{x}{x_i}}$ with $\alpha_i = \frac{A_i}{x_i}$.

The Taylor series for each function $\frac{1}{1-\frac{x}{x_i}} = \sum\limits_{n=0}^{\infty} \frac{1}{x_i^n} \cdot x^n$ about zero is a geometric series with a radius of convergence equal to $|x_i|$. Consequently, the radius of convergence for the Taylor series of $f(x)$ about zero is the minimum of the absolute values of $x_1, x_2, \ldots, x_n$, the roots of Q. Analogously, it can be shown that the radius of convergence of the Taylor series about any other point x' is equal to the minimal distance between x' and the roots of P.

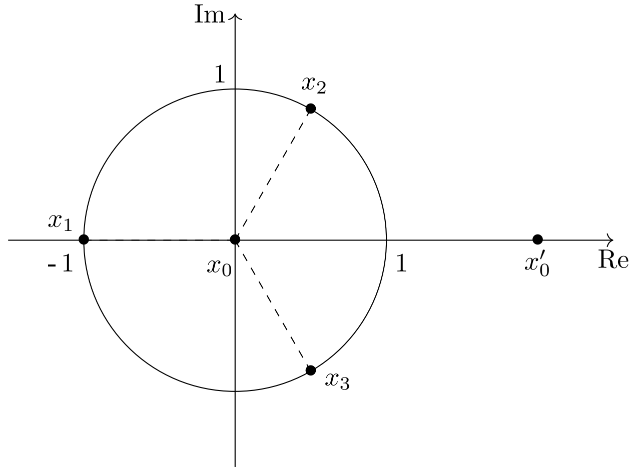
Remark. If $\deg(P) \geq \deg(Q)$, long division of polynomials allows us to express $f(x) = H(x) + \frac{\widetilde{P}(x)}{Q(x)}$, where $H(x)$ is a polynomial and $\deg(\widetilde{P}) < \deg(Q)$. Importantly, any polynomial coincides with its own Taylor series about 0 with an infinite radius of convergence. Therefore, the initial restriction on degrees doesn't result in any loss of generality, and the argument presented above is valid for any rational function.

Example. Determine a lower bound for the radius of convergence of series solutions about $x_0 = 0$ and $x_0' = 2$ for the differential equation $(1 + x^3)y'' + 4xy' + y = 0$.

We start by checking that $1 + 0^3 = 1 \neq 0$ and $1 + 2^3 = 9 \neq 0$, hence both points are regular. We rewrite the equation in the form $y'' + \frac{4x}{1+x^3} y' + \frac{1}{1+x^3} y = 0$ and notice that $p(x) = \frac{4x}{1+x^3}$ and $q(x) = \frac{1}{1+x^3}$ are rational functions with degrees of numerators smaller than the denominator. Next, we find the roots of the polynomial in the denominator. There is an obvious real root $x_1 = -1$, which allows us to factor the polynomial as $1 + x^3 = (x + 1)(x^2 - x + 1)$ and find the remaining two conjugate complex roots $x_2 = \frac{1+\sqrt{3}i}{2}$ and $x_3 = \frac{1-\sqrt{3}i}{2}$.

Since all three roots are on the unit circle inside the complex plane, the distance from $x_0 = 0$ (or $(0, 0)$ on the complex plane) to each of them is 1. This gives a lower bound for the radius of convergence of series solutions about x_0 equal to 1.

The distance between $x_0' = 2$ (or $(2, 0)$ on the complex plane) and $x_1 = (-1, 0)$ is equal to 3, while the distances from x_0' to $x_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $x_3 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ are equal to $\sqrt{\left(2 - \frac{1}{2}\right)^2 + \left(0 - \frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{3}$. We conclude that a lower bound for the radius of convergence of series solutions about x_0' is given by $\sqrt{3} = \min(3, \sqrt{3})$.



Lecture 8
Euler Equation and Regular Singular Points

In this lecture, we will explore methods to solve equations of the form

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

in the vicinity of a singular point x_0. Recall that if the polynomials P, Q, and R have no common factors, then the singular points occur where $P(x) = 0$.

Euler Equation

A relatively simple differential equation with a singular point is the Euler equation

$$x^2y'' + \alpha xy' + \beta y = 0,$$

where α and β are real numbers. The corresponding differential operator is $\mathcal{D} = x^2\frac{d^2}{dx^2} + \alpha x\frac{d}{dx} + \beta$, and a function y is a solution if and only if $\mathcal{D}(y) = 0$. Since $P(x) = x^2$, the only singular point is $x = 0$; all other points are ordinary points.

We begin by seeking solutions on the interval $x > 0$. Observe that

$$\mathcal{D}(x^r) = r(r-1)x^r + r\alpha x^r + \beta x^r = (r(r-1) + r\alpha + \beta)x^r,$$

which equals zero identically for all x if $F(r) := r(r-1) + r\alpha + \beta = r^2 + (\alpha - 1)r + \beta = 0$. Notice that $F(r)$ is a quadratic polynomial.

Now consider three cases corresponding to the sign of the discriminant $D = (\alpha - 1)^2 - 4\beta$.

1. If $D > 0$, then $F(r)$ has two distinct real roots expressed in terms of parameters α and β as

$$r_1 = \frac{(1-\alpha) + \sqrt{(\alpha-1)^2 - 4\beta}}{2} \text{ and } r_2 = \frac{(1-\alpha) - \sqrt{(\alpha-1)^2 - 4\beta}}{2}.$$

Evaluating the Wronskian

$$W(x^{r_1}, x^{r_2}) = r_2x^{r_1+r_2-1} - r_1x^{r_1+r_2-1} = (r_2 - r_1)x^{r_1+r_2-1},$$

and observing that it has only one zero at $x = 0$ and does not identically vanish on any interval around 0, allows us to conclude that the functions $\{x^{r_1}, x^{r_2}\}$ form a fundamental set of solutions in this case.

2. In case $D = 0$ (equivalently $\beta = \frac{(\alpha-1)^2}{4}$), the polynomial has a single zero at $r_1 = \frac{1-\alpha}{2}$, giving us the solution x^{r_1}. However, another solution, linearly independent from the first one, remains to be found. Inspired by the fact that r_1 is not only a zero of $F(r)$ but also of its derivative $F'(r)$, we can exploit the equality $\mathcal{D}(x^r) = F(r)x^r$ by taking derivatives of both sides with respect to r.

   This allows us to obtain
$$\mathcal{D}(\ell n(x)x^r) = F'(r)x^r + \ell n(x)F(r)x^r$$
   (we have used that the operators $\mathcal{D}$ and $\frac{\partial}{\partial r}$ commute). In our case, $F(r) = (r - r_1)^2$, so the latter equation becomes
$$\mathcal{D}(\ell n(x)x^{r_1}) = 2(r - r_1)x^{r_1} + \ell n(x)(r - r_1)^2 x^{r_1}.$$
   We conclude that $\mathcal{D}(\ell n(x)x^{r_1}) = 0$, and hence, the function $\ell n(x)x^{r_1}$ is a solution to Euler's equation. It remains to check that
$$\begin{aligned} W(x^{r_1}, \ell n(x)x^{r_1}) &= x^{r_1}(x^{r_1-1} + r_1\ell n(x)x^{r_1-1}) - r_1 x^{r_1-1}\ell n(x)x^{r_1} \\ &= x^{2r_1-1} + r_1\ell n(x)x^{2r_1-1} - r_1\ell n(x)x^{2r_1-1} = x^{2r_1-1} \end{aligned}$$
   has only one zero at $x = 0$. Therefore, the functions $\{x^{r_1}, \ell n(x)x^{r_1}\}$ form a fundamental set of solutions.

3. It remains to consider the case when $D < 0$, and the roots of $F(r)$ are conjugate complex numbers $r_1 = a + bi$ and $r_2 = a - bi$, where $b \neq 0$. By employing the equality $x^r = e^{r\ell n(x)}$ and applying transformations akin to those used to derive real solutions like $e^{ax}\cos(bx)$ and $e^{ax}\sin(bx)$ from $e^{(a+bi)x}$ and $e^{(a-bi)x}$ for linear equations of order 2 with constant coefficients and a characteristic polynomial featuring a negative determinant, we can derive the solutions $x^a\cos(b\ell n(x))$ and $x^a\sin(b\ell n(x))$ from $e^{r_1 x} = e^{(a+bi)x}$ and $e^{r_2 x} = e^{(a-bi)x}$ (see Chapter 5.4 of the book for details).

   It is straightforward to check that the Wronskian is not identically zero within any interval around zero. Consequently, these solutions are linearly independent.

Let's consider an example.

Example. Find the general solution of the equation $x^2y'' - 3xy' + 4y = 0$ on the interval $x > 0$. This equation has $\alpha = -3, \beta = 4$, yielding $F(r) = r^2 - 4r + 4 = (r-2)^2$ with a single root $r = 2$. Hence, we are in the second case, and we obtain the fundamental solutions $y_1 = x^2$ and $y_2 = x^2\ell n(x)$. Let's verify that the function $y = x^2\ell n(x)$ is indeed a solution of the equation $x^2y'' - 3xy' + 4y = 0$. We compute $y' = 2x\ell n(x) + x$ and $y'' = 2\ell n(x) + 2 + 1 = 2\ell n(x) + 3$, giving
$$\begin{aligned} x^2y'' - 3xy' + 4y &= x^2(2\ell n(x) + 3) - 3x(2x\ell n(x) + x) + 4x^2\ell n(x) \\ &= 2x^2\ell n(x) + 3x^2 - 6x^2\ell n(x) - 6x^2 - 3x^2 + 4x^2\ell n(x) = 0 \quad \checkmark \end{aligned}$$

Euler equation: $x < 0$

Thus far, we have successfully solved Euler equation on the open interval $x > 0$ (notice that, for instance, $\ell n(x)$ is not defined for $x \leq 0$). To obtain real-valued solutions of the equation in the interval $x < 0$, we can reduce it to the case with $\widetilde{x} > 0$ simply by making the change of variable $x = -\widetilde{x}$. Using the chain rule, we find $\frac{dy}{dx} = \frac{dy}{dx}\frac{dy}{d\widetilde{x}} = -\frac{dy}{d\widetilde{x}}$, and, similarly, $\frac{d^2y}{dx^2} = \frac{d^2y}{d\widetilde{x}^2}$. Hence, the original equation $x^2y'' + \alpha xy' + \beta y = 0$ with $x < 0$ becomes $(-\widetilde{x})^2y'' + \alpha(-\widetilde{x})(-y') + \beta y = 0 \Leftrightarrow \widetilde{x}^2y'' + \alpha\widetilde{x}y' + \beta y = 0$, where now y is a function of $\widetilde{x}$ with $\widetilde{x} > 0$. The transformed equation is exactly the same as the original one. This allows us to extend the results that we have already obtained for positive values of x to any interval not containing the origin: If the roots are real and different, then the general solution is
$$y = c_1|x|^{r_1} + c_2|x|^{r_2}.$$
If the roots are real and equal, then the general solution is
$$y = (c_1 + c_2\ell n|x|)|x|^{r_1}.$$

If the roots are complex conjugates, then the general solution is

$$y = |x|^a [c_1 \cos(b\ell n|x|) + c_2 \sin(b\ell n|x|)].$$

We now bring our attention back to the general equation

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

in the vicinity of a singular point x_0. This means that $P(x_0) = 0$, and at least one of Q and R is not zero at x_0.

It is noteworthy that solutions of Euler's equation included the functions $\ell n(x)$ and x^a with $a < 0$, which are not analytic at 0 and, consequently, cannot be represented by a Taylor series in powers of x, rendering the methods of the previous lecture inapplicable. This behavior is typical for singular points.

A natural question at this point is the following.

Question. When faced with the intricate nature of singular points in differential equations, one might wonder: Why bother? why not simply disregard them?

Singularities, though rare, play a crucial role in understanding the essence of differential equations. Ignoring them means sidestepping key junctures where our theories confront challenges. Consider black holes—stellar anomalies with singularities at their core. These points unveil uncharted territories where classical physics falls short.

Despite their complexity, the study of singularities, championed by thinkers like Hawking and Penrose, proves indispensable. Their insights deepen our grasp of extreme phenomena and push us to develop more robust theories capable of navigating the mysterious corners of the mathematical universe.

Singularities: how bad do they get?

Singular points are classified into two types: regular and irregular. This categorization aids in understanding the behavior of solutions in the vicinity of these points.

Singular point x_0

Regular

$\lim\limits_{x\to x_0} (x-x_0)\frac{Q(x)}{P(x)}$ is finite
$\lim\limits_{x\to x_0} (x-x_0)^2\frac{R(x)}{P(x)}$ is finite
$(x-x_0)\frac{Q(x)}{P(x)}$ and $\lim\limits_{x\to x_0} (x-x_0)^2\frac{R(x)}{P(x)}$
are analytic at $x = x_0$

Irregular

otherwise

Example. Let's determine the type of the singularity in Euler equation $x^2y'' + \alpha xy' + \beta y = 0$ at $x_0 = 0$. Computing the limits,

$$\lim_{x\to 0} x \cdot \frac{\alpha x}{x^2} = \alpha,$$
$$\lim_{x\to 0} x^2 \cdot \frac{\beta}{x^2} = \beta,$$

we conclude that the singularity is regular.

We will observe that solutions near a regular singular point exhibit similarities with those for Euler equations near zero. Specifically, solutions in the vicinity of a regular singular point may involve powers of x with negative or nonintegral exponents, logarithmic functions, or trigonometric functions of logarithmic arguments. Our focus will be on elucidating the techniques for solving second-order linear ODEs in the proximity of a regular singular point. The analysis of solutions around irregular singular points entails a more intricate discussion, which lies beyond the scope of this course.

Example. Find singular points of the given equations and determine their type.

1. $(x+2)^2(x-1)y''+3(x-1)y'-2(x+2)y$. The singular points are zeros of the polynomial $(x+2)^2(x-1)$, which are $x_0=-2$ and $x_1=1$. We compute

$$\begin{aligned}\lim_{x\to-2}(x+2)\cdot\frac{3(x-1)}{(x+2)^2(x-1)} &= \lim_{x\to-2}\frac{3}{(x+2)}, \text{DNE}\\ \lim_{x\to1}(x-1)\cdot\frac{3(x-1)}{(x+2)^2(x-1)} &= \lim_{x\to1}\frac{3(x-1)}{(x+2)^2}=0,\\ \lim_{x\to1}(x-1)^2\cdot\frac{-2(x+2)}{(x+2)^2(x-1)} &= \lim_{x\to1}(x-1)\cdot\frac{-2}{(x+2)}=0,\end{aligned}$$

   and conclude that x_0 is an irregular singularity, while x_1 is regular.

2. $xy''+e^xy'+3\cos(x)y=0$. The only singular point is $x=0$. We compute

$$\begin{aligned}\lim_{x\to0}x\cdot\frac{e^x}{x} &= \lim_{x\to0}e^x=1\\ \lim_{x\to0}x^2\cdot\frac{3\cos(x)}{x} &= \lim_{x\to0}3x\cos(x)=0.\end{aligned}$$

   As the functions e^x and $x\cos(x)$ are analytic at 0 conclude that $x_0=0$ is a regular singularity.

Lecture 9
Series Solutions Near a Regular Singular Point

Consider the problem of solving the general second-order linear equation

$$P(x)y''+Q(x)y'+R(x)y=0$$

in the vicinity of a regular singular point $x=x_0$. For convenience, we assume $x_0=0$. If $x_0\neq0$, a simple linear transformation $t=x-x_0$ can be applied to transform the equation into one with the regular singular point at the origin.

Recall that the assumption of $x=0$ being a regular singular point means that both limits $\lim\limits_{x\to0}\frac{xQ(x)}{P(x)}$ and $\lim\limits_{x\to0}\frac{x^2R(x)}{P(x)}$ are finite, and the functions $q(x):=\frac{xQ(x)}{P(x)}$ and $r(x):=\frac{x^2R(x)}{P(x)}$ are analytic at $x=0$.

Multiplying both sides of the equation $P(x)y''+Q(x)y'+R(x)y=0$ by $\frac{x^2}{P(x)}$, we obtain $x^2y''+xq(x)y'+r(x)y=0$.

Remark. In the special case where all coefficients $q_n, n>0$, and $r_n, n>0$ are zero, i.e., the functions $q(x)=q_0$ and $r(x)=r_0$ are constant on some interval around zero, we obtain Euler's equation $x^2y''+xq_0y'+r_0y=0$.

We seek a series solution in the form

$$y=x^s\sum_{n=0}^{\infty}c_nx^n=\sum_{n=0}^{\infty}c_nx^{n+s},$$

where the integer(s) s, for which the solutions in the given form exist, are to be determined in the process.

Example. Let's explore the equation $3x^2y'' + 2xy' + x^2y = 0$. We employ the series solution $y = \sum\limits_{n=0}^{\infty} c_n x^{n+s}$ along with its derivatives $y' = \sum\limits_{n=0}^{\infty}(n+s)c_n x^{n+s-1}$ and $y'' = \sum\limits_{n=0}^{\infty}(n+s)(n+s-1)c_n x^{n+s-2}$.

Substituting these into the differential equation, we get:

$$\begin{aligned}
&3x^2\sum_{n=0}^{\infty}(n+s)(n+s-1)c_n x^{n+s-2} + 2x\sum_{n=0}^{\infty}(n+s)c_n x^{n+s-1} + x^2\sum_{n=0}^{\infty}c_n x^{n+s} \\
&= 3\sum_{n=0}^{\infty}(n+s)(n+s-1)c_n x^{n+s} + 2\sum_{n=0}^{\infty}(n+s)c_n x^{n+s} + \sum_{n=0}^{\infty}c_n x^{n+s+2} = 0
\end{aligned}$$

The latter equality must hold true for all values of x within the interval under consideration and, therefore, imposes a system of equations on coefficients of powers of x, setting each of them equal to zero. The first such equation, in our case,

$$(3s(s-1) + 2s)c_0 = 0 \Leftrightarrow (3s^2 - s)c_0 = 0,$$

is referred to as the **indicial equation**. We assume that $c_0 \neq 0$, limiting the feasible values for s to $s = 0$ and $s = \frac{1}{3}$.

Definition. The roots of the indicial equation are called the **exponents at the singularity** for the regular singular point $x = 0$.

The coefficient in front of x^{s+1} equals $(3s(1+s) + 2(1+s))c_1 = 0$. Since neither $s = 0$ nor $s = \frac{1}{3}$ satisfies $3s^2+5s+2 = 0$, we are forced to set $c_1 = 0$. The remaining equations, $(3(n+s)(n+s-1)+2(n+s))c_n + c_{n-2} = 0$ with $n \geq 2$, allow us to express

$$c_n = -\frac{c_{n-2}}{3(n+s)(n+s-1) + 2(n+s)} = \frac{c_{n-2}}{(n+s)(1-3n-3s)}.$$

It follows that coefficients with odd indices vanish (they are all proportional to c_1), while those with even indices can be recursively expressed in terms of c_0. Plugging in $s = 0$, we get

$$c_n = \begin{cases} \frac{c_{n-2}}{n-3n^2} = \frac{c_{n-2}}{n(1-3n)}, & n \text{ is even} \\ 0, & n \text{ is odd.} \end{cases}$$

As n is even, we can write $n = 2m$ for m a positive integer. Then

$$\begin{aligned}
c_{2m} &= \frac{c_{2m-2}}{2m(1-6m)} = \frac{c_{2m-4}}{2m(1-6m)(2m-2)(1-6(m-1))} = \frac{c_{2m-4}}{2m(1-6m)(2m-2)(7-6m)} \\
&= \frac{c_{2m-4}}{2m(2m-2)(1-6m)(7-6m)} = \ldots = \frac{c_0}{2m(2m-2)(2m-4)\ldots 2\cdot(1-6m)(7-6m)(13-6m)\ldots 1},
\end{aligned}$$

where the rightmost 1 in the denominator comes from $1 = 1 + 6\cdot m - 6m$. Further simplifying the denominator gives

$$2m(2m-2)(2m-4)\ldots 2\cdot(1-6m)(7-6m)(13-6m)\ldots 1 = 2^m m(m-1)(m-2)\ldots 1\cdot(1-6m)(7-6m)(13-6m)\ldots 1$$

This yields the formula

$$c_{2m} = \frac{c_0}{2^m m! \cdot \prod\limits_{\ell=0}^{m}(1 + 6(\ell - m))}.$$

The power series solution y_1 corresponding to $c_0 = 1$ (and $s = 0$) is

$$y_1 = \sum_{n=0}^{\infty}\frac{x^{2n}}{2^n n! \cdot \prod\limits_{\ell=0}^{n}(1 + 6(\ell - n))}.$$

Finding the solution y_2 from $s = \frac{1}{3}$ is analogous (but even slightly more annoying 🙂).

This example highlights an approach to determine power series solutions in the vicinity of a regular singular point.

In general, when dealing with the equation $P(x)y'' + Q(x)y' + R(x)y = 0$ near a regular singularity, finding a fundamental set $\{y_1, y_2\}$ of solutions follows a similar approach to the example we have discussed. If the indicial equation has two distinct roots s_1 and s_2 where $s_1 - s_2$ is not an integer, we typically obtain two linearly independent solutions of the form $y = \sum\limits_{n\geq 0} c_n x^{n+s}$. When the roots of the indicial equation are conjugate complex numbers, their difference cannot be an integer. Consequently, we always have two solutions, albeit complex-valued. However, akin to the Euler equation, we can derive real-valued solutions by extracting the real and imaginary parts of the complex solutions.

On the other hand, if the roots of the indicial equation are equal or differ by an integer, we usually find one solution in the form $y = \sum\limits_{n\geq 0} c_n x^{n+s}$, while the second solution tends to exhibit a more intricate structure.

Lecture 10

A Quantum of Boundary Value Problems and Linear Algebra

Up to this point, our focus has been on initial value problems, characterized by a differential equation coupled with appropriate initial conditions at a specific point. A classic instance is the differential equation:

$$y'' + p(t)y' + q(t)y = F(x)$$

accompanied by the initial conditions:

$$y(t_0) = y_0, \quad y'(t_0) = y_0'.$$

In practical applications, scenarios often arise where either the value of the dependent variable y or its derivative is specified at two distinct points. These specifications are termed *boundary conditions*, distinguishing them from initial conditions that pertain to the value of y and y' at the same point. Consequently, a differential equation, when paired with suitable boundary conditions, constitutes a two-point boundary value problem.

Boundary Value Problems

Consider the following typical example:

$$y'' + p(x)y' + q(x)y = F(x)$$

with the boundary conditions:

$$y(\alpha) = y_0, \quad y(\beta) = y_1.$$

It is noteworthy that boundary value problems commonly involve a spatial coordinate as the independent variable, hence the use of x rather than t.

To solve the two-point boundary value problem, our objective is to find a function $y = \varphi(x)$ that not only satisfies the differential equation within the interval $\alpha < x < \beta$ but also takes on the prescribed values y_0 and y_1 at the endpoints of the interval. Typically, the approach involves seeking the general solution of the differential equation first, and subsequently employing the boundary conditions to determine the values of the arbitrary constants.

If the function $F(x) = 0$ and the boundary values y_0 and y_1 are also zero, we categorize the problem as homogeneous. Otherwise, when $F(x)$ is non-zero, or if the boundary values y_0 and y_1 are non-zero, the problem is deemed nonhomogeneous.

While boundary value problems may bear a resemblance to initial value problems, their solutions exhibit crucial distinctions. With mild conditions on the coefficients, initial value problems are guaranteed to possess a unique solution. In contrast, boundary value problems, even under comparable conditions, may yield a unique solution, remain unsolvable, or, intriguingly, yield infinitely many solutions. This nuanced behavior parallels systems of linear algebraic equations, emphasizing the distinctive nature of linear boundary value problems.

Let's take a look at some examples.

Example. 1. Consider the differential equation $x^2y''-2xy'+2y=0$ with boundary conditions $y(1)=-1$ and $y(2)=1$. This equation is an Euler equation with parameters $\alpha=-2$ and $\beta=2$. We compute $F(r)=r^2-3r+2=(r-1)(r-2)$, concluding that the general solution is $y=C_1x+C_2x^2$. The boundary conditions impose the following system of equations on C_1 and C_2:

$$\begin{cases} y(1)=C_1+C_2=-1 \\ y(2)=2C_1+4C_2=1 \end{cases}$$

Solving this system yields $C_1=-2.5$ and $C_2=1.5$. Hence the solution is $y=-2.5x+1.5x^2$.

2. Next, let's tackle $y''+4y=\cos(x)$ with boundary conditions $y(0)=0$ and $y(\pi)=0$. We first solve the complementary equation $y''+4y=0$ to obtain $y_c=C_1\cos(2x)+C_2\sin(2x)$. Employing the method of undetermined coefficients with $y_p=A\cos(x)$, we find a particular solution of the initial equation $y_p=\frac{1}{3}\cos(x)$. Hence, the general solution of the equation is $y=y_c+y_p=C_1\cos(2x)+C_2\sin(2x)+\frac{1}{3}\cos(x)$. The boundary conditions impose the following system of equations on C_1 and C_2:

$$\begin{cases} y(0)=C_1+\frac{1}{3}=0 \\ y(\pi)=C_1-\frac{1}{3}=0, \end{cases}$$

which has no solutions.

We observe that after finding the general solution y, the boundary values impose linear equations on the unknown values of y at certain points, i.e. our problem reduces to one from linear algebra.

Eigenvalues, Eigenvectors, and Eigenfunctions

Recall that in linear algebra, a vector $(v_1,\ldots,v_n)$ that satisfies the matrix equation $A\cdot v=\lambda v$ for some scalar λ is termed an *eigenvector*, while λ is referred to as an *eigenvalue*. Likewise, in the context of differential operators, a function y satisfying $\mathcal{D}(y)=\lambda y$ (where $\mathcal{D}$ is an operator and λ is a constant) is termed an *eigenfunction* of $\mathcal{D}$.

It's noteworthy that David Hilbert was the first to employ the German term *eigen*, meaning "own," to denote eigenvalues and eigenvectors. This terminology became widely adopted in mathematics. However, it's conceivable that Hilbert was influenced by a similar usage by Hermann von Helmholtz.

Example. Consider the equation $y''=\lambda y$. The equation $y''-\lambda y=0$ is a homogeneous equation of order 2 with constant coefficients and has a fundamental set of solutions $\{e^{-\sqrt{\lambda}x},e^{\sqrt{\lambda}x}\}$. Therefore, one has a two-dimensional space of eigenfunctions with basis $\langle e^{-\sqrt{\lambda}x},e^{\sqrt{\lambda}x}\rangle$ for each eigenvalue $0\neq\lambda\in\mathbb{R}$ and $\langle 1,x\rangle$ for $\lambda=0$.

Example. Consider the differential equation $Ay''=By$, where $A<0$ and $B>0$, subject to the boundary conditions $y(0)=y(L)=0$. This equation simplifies to $y''-\frac{B}{A}y=0$. Denoting $\lambda=-\frac{B}{A}>0$, we find the characteristic polynomial $r^2+\lambda$ with roots $r_1=-i\sqrt{\lambda}$ and $r_2=i\sqrt{\lambda}$. The general solution is then $y=C_1\cos(\sqrt{\lambda}x)+C_2\sin(\sqrt{\lambda}x)$.

Next, we seek a particular solution satisfying the boundary conditions:

$$\begin{cases} C_1\cos(\sqrt{\lambda}\cdot 0)+C_2\sin(\sqrt{\lambda}\cdot 0)=0 \\ C_1\cos(\sqrt{\lambda}\cdot L)+C_2\sin(\sqrt{\lambda}\cdot L)=0 \end{cases}$$

By solving, we determine $C_1=0$ and $C_2\sin(\sqrt{\lambda}\cdot L)=0$. We notice that $\sin(\sqrt{\lambda}\cdot L)=0$ if and only if $\sqrt{\lambda}\cdot L=\pi k$ for some $k\in\mathbb{Z}$. Since $\sin(-kx)=-\sin(kx)$, we only need to consider $k\in\mathbb{Z}_{>0}$. Therefore, if λ and L satisfy $\sqrt{\lambda}\cdot L=\pi k$, then we obtain a family of solutions $y=C_2\sin(\sqrt{\lambda}x)$; otherwise, if $C_2=0$, the only solution is trivial, $y=0$.

Free Particle in an Open Box (optional)

Let's enrich our mathematical exploration with a physical context. Suppose we have a particle confined to move along the interval $[0, L]$. Considering $A = -\frac{\hbar^2}{2m}$, where m is the mass of the particle and $\hbar$ represents the Planck constant, and $B = E$ as a parameter denoting the energy of the particle, we arrive at the time-independent Schrödinger equation for a free particle on a line: $-\frac{\hbar^2}{2m}y'' = Ey$. The differential operator $H = -\frac{\hbar}{2m}\frac{d^2}{dx^2}$ is called the *Hamiltonian*, representing the total energy of the system and playing a central role in quantum mechanics. The eigenfunctions of H are the *wave functions*, which describe the possible states of the particle. Our boundary conditions confine the particle's position to be between 0 and L (referred to as "in a box").

Expanding on our previous results, $\sqrt{\lambda} \cdot L = \pi k, \quad k \in \mathbb{Z}_{>0}$ yields

$$\sqrt{\frac{2mE}{\hbar}} \cdot L = \pi k, \quad k \in \mathbb{Z}_{>0} \Leftrightarrow E_k = \frac{\hbar^2 \pi^2 k^2}{2mL^2}, \quad k \in \mathbb{Z}_{>0}.$$

In the latter equation, E is the only variable, and the values of E that satisfy the equation give rise to nontrivial solutions. For instance, $E_1 = \frac{\hbar^2\pi^2}{2mL^2}$ and $E_2 = \frac{4\hbar^2\pi^2}{2mL^2}$. The solutions to the corresponding equations $-\frac{\hbar^2}{2m}y'' = E_k y$ are known as wave functions and are traditionally represented by the Greek letter Ψ; thus, the solution to $-\frac{\hbar^2}{2m}y'' = E_k y$ would be denoted by $\Psi_k(x)$.

You may have noticed that the constant C_2 seems to have "vanished" and does not appear in the wave function $\Psi_k(x)$. This omission arises because the wave function must satisfy an additional requirement: $\int_0^L |\Psi_k(x)|^2\, dx = 1$. This condition arises from the fact that the integral represents the probability of the particle being somewhere within the interval $[0, L]$, which, according to the initial assumption, is equal to 1.

Quick Reminder on Linear Algebra

Let's quickly recap some fundamental concepts of linear algebra. Consider a vector space V over $\mathbb{R}$ equipped with an inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$. This inner product satisfies the following properties:

- Linearity: For any vectors $v, w, h \in V$ and scalars $a, b \in \mathbb{R}$, we have $\langle av + bw, h\rangle = a\langle v, h\rangle + b\langle w, h\rangle$.
- Symmetry: $\langle v, w\rangle = \langle w, v\rangle$ for all $v, w \in V$.
- Positive Definiteness: $\langle v, v\rangle \geq 0$ for all $v \in V$, with equality if and only if $v = 0$.

An inner product not only defines how vectors interact with each other but also gives us the notion of length in V. For instance, the length (or magnitude) of a vector v is given by $|v| := \sqrt{\langle v, v\rangle}$.

A *basis* for V is a set of linearly independent vectors that span the space. An *orthonormal basis* is a set of vectors $e_1, e_2, \ldots, e_n$ in which each vector has unit length and is orthogonal to all other vectors in the basis. In terms of the inner product, this means that for any two vectors e_i and e_j in the basis, we have

$$\langle e_i, e_j\rangle = \begin{cases} 1, \ i = j \\ 0, \text{ otherwise.} \end{cases}$$

For example, consider the finite-dimensional vector space $V = \mathbb{R}^n$ equipped with the dot product. Let $e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$ be the standard basis. Then the dot product is given by the formula $\langle v, w\rangle = a_1b_1 + a_2b_2 + \ldots + a_nb_n$, where $v = a_1e_1 + a_2e_2 + \ldots + a_ne_n$ and $w = b_1e_1 + b_2e_2 + \ldots + b_ne_n$. As $(e_i, e_i) = 1$ and $(e_i, e_j) = 0$ if $i \neq j$, the basis is orthonormal with respect to the dot product

To express a vector v as a linear combination of the orthonormal basis vectors, $v = a_1e_1 + a_2e_2 + \ldots + a_ne_n$, we compute the coefficients a_i using the formula $a_i := \langle v, e_i\rangle$. Similarly, to express v as a linear combination of the orthogonal basis vectors, $v = a_1e_1 + a_2e_2 + \ldots + a_ne_n$, we calculate the coefficients a_i using the formula $a_i := \frac{\langle v, e_i\rangle}{\langle e_i, e_i\rangle}$. This formula ensures that each coefficient a_i represents the projection of v onto the direction of the corresponding basis vector e_i, scaled by the length of that vector. Thus, the coefficients a_i provide the coordinates of the vector v in the orthogonal basis.

Remark. A basis is called *orthogonal* if the vectors are pairwise orthogonal, meaning $\langle e_i, e_j \rangle = 0$ for $i \neq j$, but not necessarily each vector has unit length. In order to obtain an orthonormal basis from an orthogonal one, each vector needs to be normalized by dividing it by its length: $e_i' = \frac{e_i}{|e_i|}$.

Lecture 11
Fourier Series

Fourier series are an incredibly useful tool in mathematics and engineering for representing periodic functions as a sum of sine and cosine functions. They have applications in various fields such as signal processing, image analysis, and differential equations.

Real-valued functions form a vector space over the real numbers. Unlike the finite-dimensional example from the previous section, this vector space is infinite-dimensional. Functions that are periodic with a period of $2L$ form a subspace denoted by $\mathcal{P}$. This means that any function $f(x)$ in this subspace satisfies the condition $f(x+2L) = f(x)$ for all x (in the domain). However, it is important to note that $2L$ does not necessarily have to be the smallest number for which this condition holds.

An inner product on $\mathcal{P}$ is given by

$$\langle f, g \rangle := \int_{-L}^{L} f(x)g(x)\, dx.$$

This inner product allows us to define notions of orthogonality, projection, and distance in $\mathcal{P}$.

Consider the set of functions $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}$ and $\left\{\cos\left(\frac{m\pi x}{L}\right)\right\}$, where n and m are nonnegative integers. These functions have period $T = \frac{2\pi L}{n\pi} = \frac{2L}{n}$, hence they are $2L$-periodic and belong to $\mathcal{P}$. Next, we will demonstrate that they are pairwise orthogonal with respect to our inner product:

$$\left\langle \cos\left(\frac{m\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right) \right\rangle = \begin{cases} L, & \text{if } m = n, \\ 0, & \text{otherwise,} \end{cases}$$

$$\left\langle \sin\left(\frac{m\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right) \right\rangle = \begin{cases} L, & \text{if } m = n, \\ 0, & \text{otherwise,} \end{cases}$$

$$\left\langle \cos\left(\frac{m\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right) \right\rangle = 0.$$

Let's compute some inner products explicitly:

$$\left\langle \cos\left(\frac{m\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right) \right\rangle = \int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{2} \int_{-L}^{L} \cos\left(\frac{(m-n)\pi x}{L}\right) + \cos\left(\frac{(m+n)\pi x}{L}\right) dx$$

$$= \frac{1}{2} \left(\frac{L}{\pi(m-n)} \sin\left(\frac{(m-n)\pi x}{L}\right)\Bigg|_{-L}^{L} + \frac{L}{\pi(m+n)} \sin\left(\frac{(m+n)\pi x}{L}\right)\Bigg|_{-L}^{L} \right) = 0.$$

Here we have used the formulas $\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$ allowing us to express $\cos(a)\cos(b) = \frac{\cos(a+b)+\cos(a-b)}{2}$. The remaining inner products can be computed in a similar manner to demonstrate the orthogonality of the corresponding functions.

Therefore, the functions $\mathcal{F} := \{1\} \cup \left\{\sin\left(\frac{n\pi x}{L}\right)\right\} \cup \left\{\cos\left(\frac{m\pi x}{L}\right)\right\}$ with $n, m \in \mathbb{Z}_{>0}$ form an orthogonal set of functions in $\mathcal{P}$.

However, upon evaluation, we find that

$$\left\langle \cos\left(\frac{m\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right)\right\rangle = \int_{-L}^{L} \cos^2\left(\frac{m\pi x}{L}\right)\,dx = \frac{1}{2}\int_{-L}^{L}\left(\cos\left(\frac{2m\pi x}{L}\right)+1\right)\,dx$$

$$= \frac{1}{2}\left(\frac{L}{2\pi m}\sin\left(\frac{2m\pi x}{L}\right)+x\right)\Big|_{-L}^{L} = \frac{L-(-L)}{2} = L.$$

Similarly, similarly $\left\langle \sin\left(\frac{m\pi x}{L}\right), \sin\left(\frac{m\pi x}{L}\right)\right\rangle = L$ as well. To ensure that the set of functions is orthonormal, we have two options. First, we can follow the procedure outlined in the previous section, which yields:

$$\widetilde{\mathcal{F}} := \left\{\frac{1}{2L}\right\} \cup \left\{\frac{\sin\left(\frac{n\pi x}{L}\right)}{\sqrt{L}}\right\} \cup \left\{\frac{\cos\left(\frac{m\pi x}{L}\right)}{\sqrt{L}}\right\}$$

Alternatively, we can redefine the inner product as

$$\langle f, g\rangle := \frac{1}{L}\int_{-L}^{L} f(x)g(x)\,dx.$$

For simplicity, we will opt for the latter approach.

Remark. The first function in $\mathcal{F}$ is the constant function $f(x) = 1$. Its inner product with itself is calculated as follows:

$$\langle 1, 1\rangle = \frac{1}{L}\int_{-L}^{L} 1\cdot 1\,dx = \frac{1}{L}\int_{-L}^{L} dx = 2,$$

so the function has a norm of 2.

Fourier Series: Let's Get Serious

Let $g(x)$ be a $2L$-periodic function. By shifting the x-coordinate (if necessary), we may assume that $g(-L) = g(L)$. Our goal is to express $g(x)$ in terms of the functions in $\mathcal{F}$:

$$g(x) = \frac{a_0}{2} + \sum_{n\geq 1} a_n \sin\left(\frac{n\pi x}{L}\right) + \sum_{n\geq 1} b_n \cos\left(\frac{n\pi x}{L}\right),$$

where the 2 in the denominator comes from the normalization explained in the remark above. The coefficients are given by

$$a_0 = \langle g(x), 1\rangle = \frac{1}{L}\int_{-L}^{L} g(x)\,dx,$$

$$a_n = \left\langle g(x), \sin\left(\frac{n\pi x}{L}\right)\right\rangle = \frac{1}{L}\int_{-L}^{L} g(x)\sin\left(\frac{n\pi x}{L}\right)\,dx, \quad n > 0,$$

$$b_n = \left\langle g(x), \cos\left(\frac{n\pi x}{L}\right)\right\rangle = \frac{1}{L}\int_{-L}^{L} g(x)\cos\left(\frac{n\pi x}{L}\right)\,dx, \quad n > 0.$$

Example. Find Fourier series for the following functions.

1. $f(x) = \begin{cases} 1, & -L \leq x < 0, \\ 0, & 0 \leq x < L; \end{cases} \qquad f(x+2L) = f(x)$

To find the Fourier series for the given piecewise function $f(x)$, we begin by determining the coefficients a_0, a_n, and b_n.

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_{-L}^{0} dx = \frac{1}{L}\cdot x\Big|_{-L}^{0} = (0-(-L)) = \frac{1}{L}\cdot(0-(-L)) = 1.$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx = \frac{1}{L}\int_{-L}^{0} \sin\left(\frac{n\pi x}{L}\right)\,dx = -\frac{L}{n\pi L}\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{0} = -\frac{1}{n\pi}(1-\cos(n\pi)).$$

We conclude that:

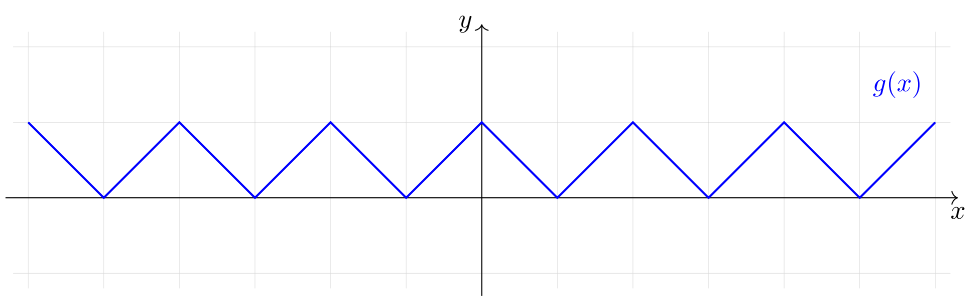
$$a_n = \begin{cases} -\frac{2}{(2m+1)\pi}, & n = 2m+1 \text{ is odd,} \\ 0, & n \text{ is even.} \end{cases}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx = \frac{1}{L}\int_{-L}^{0} \cos\left(\frac{n\pi x}{L}\right)\,dx = \frac{L}{n\pi L}\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{0} = \frac{1}{n\pi}(0-0) = 0.$$

Therefore, the Fourier series for $f(x)$ is

$$f(x) = \frac{1}{2} - \frac{2}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)}\sin\left(\frac{(2m+1)\pi x}{L}\right).$$

2. $g(x) = \begin{cases} x+1, & -1 \le x < 0, \\ 1-x, & 0 \le x < 1; \end{cases} \qquad g(x+2) = g(x)$



In order to find the Fourier series for $g(x)$, we begin by determining the coefficients a_0, a_n, and b_n.

$$a_0 = \int_{-1}^{1} g(x)\,dx = \int_{-1}^{0}(x+1)\,dx + \int_{0}^{1}(1-x)\,dx = \left(\frac{x^2}{2}+x\right)\Big|_{-1}^{0} + \left(x-\frac{x^2}{2}\right)\Big|_{0}^{1} = 1.$$

$$b_n = \int_{-1}^{1} g(x)\cos\left(\frac{n\pi x}{1}\right)\,dx = \left(\int_{-1}^{0}(x+1)\cos(n\pi x)\,dx + \int_{0}^{1}(1-x)\cos(n\pi x)\,dx\right).$$

First, we compute the general antiderivatives.

$$\int \cos(n\pi x)\,dx = \frac{\sin(n\pi x)}{n\pi} + C.$$

To find the antiderivative of $x\cos(n\pi x)$, we employ integration by parts. Let $u = x$ and $dv = \cos(n\pi x)\,dx$. Then, we differentiate u to get $du = dx$ and integrate dv to get $v = \frac{\sin(n\pi x)}{n\pi}$. The integration by parts formula gives

$$\int x\cos(n\pi x)\,dx = x\left(\frac{\sin(n\pi x)}{n\pi}\right) - \int \frac{\sin(n\pi x)}{n\pi}\,dx = \frac{x\sin(n\pi x)}{n\pi} - \frac{1}{n\pi}\int \sin(n\pi x)\,dx$$
$$= \frac{x\sin(n\pi x)}{n\pi} + \frac{1}{n\pi}\cdot\frac{\cos(n\pi x)}{n\pi} + C = \frac{x\sin(n\pi x)}{n\pi} + \frac{\cos(n\pi x)}{n^2\pi^2} + C.$$

This allows to obtain

$$b_n = \int_{-1}^{0}(x+1)\cos(n\pi x)\,dx + \int_{0}^{1}(1-x)\cos(n\pi x)\,dx = \left(\frac{x\sin(n\pi x)}{n\pi} + \frac{\cos(n\pi x)}{n^2\pi^2} + \frac{\sin(n\pi x)}{n\pi}\right)\Bigg|_{-1}^{0}$$
$$\left(\frac{\sin(n\pi x)}{n\pi} - \frac{x\sin(n\pi x)}{n\pi} - \frac{\cos(n\pi x)}{n^2\pi^2}\right)\Bigg|_{0}^{1} = \left(\frac{1}{n^2\pi^2} - \frac{\cos(-n\pi)}{n^2\pi^2}\right) + \left(-\frac{\cos(n\pi)}{n^2\pi^2} + \frac{1}{n^2\pi^2}\right) = \frac{2(1-\cos(n\pi))}{n^2\pi^2}.$$

We conclude that:

$$b_n = \begin{cases} \frac{4}{\pi^2(2m+1)^2}, & n = 2m+1 \text{ is odd},\\ 0, & n \text{ is even}.\end{cases}$$

It remains to find the coefficients

$$a_n = \int_{-1}^{1} g(x)\sin\left(\frac{n\pi x}{1}\right)\,dx = \left(\int_{-1}^{0}(x+1)\sin(n\pi x)\,dx + \int_{0}^{1}(1-x)\sin(n\pi x)\,dx\right).$$

Similarly to how we did before, we find $\int \sin(n\pi x)\,dx = -\frac{\cos(n\pi x)}{n\pi} + C$ and $\int x\sin(n\pi x)\,dx = -\frac{x\cos(n\pi x)}{n\pi} + \frac{\sin(n\pi x)}{n^2\pi^2} + C$ to come up with

$$a_n = \int_{-1}^{0}(x+1)\sin(n\pi x)\,dx + \int_{0}^{1}(1-x)\sin(n\pi x)\,dx = \left(-\frac{x\cos(n\pi x)}{n\pi} + \frac{\sin(n\pi x)}{n^2\pi^2} - \frac{\cos(n\pi x)}{n\pi}\right)\Bigg|_{-1}^{0}$$
$$+\left(-\frac{\cos(n\pi x)}{n\pi} + \frac{x\cos(n\pi x)}{n\pi} - \frac{\sin(n\pi x)}{n^2\pi^2}\right)\Bigg|_{0}^{1} = \left(-\frac{1}{n\pi} - \frac{\cos(-n\pi)}{n\pi} + \frac{\cos(-n\pi)}{n\pi}\right)$$
$$+\left(-\frac{\cos(n\pi)}{n\pi} + \frac{\cos(n\pi)}{n\pi} + \frac{1}{n\pi}\right) = 0.$$

The Fourier series for $g(x)$ is

$$g(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\cos\left((2m+1)\pi x\right).$$

Observe that the function $g(x)$ from the previous example is even, as $g(-x) = 1-(-x) = 1+x = g(x)$. Consequently, the Fourier series of $g(x)$ consisted only of cosine terms. This observation holds more generally.

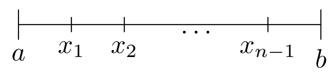
Remark. If a periodic function $f(x)$ is odd, its Fourier series will comprise only sine terms. This outcome arises from the cancellation of cosine terms due to the fact that the integral of an odd function over a domain symmetric with respect to the y-axis is zero. Additionally, when multiplied by cosine functions, the resulting product $f(x)\cos\left(\frac{n\pi x}{L}\right)$ is odd, leading to a vanishing contribution from cosine terms.

Conversely, if $f(x)$ is even, then each function $f(x)\sin\left(\frac{n\pi x}{L}\right)$ is odd. Therefore, its Fourier series will consist solely of cosine terms.

In the study of periodic functions, one fundamental question arises: under what conditions can a periodic function be expressed as a Fourier series? To address this inquiry, we introduce the notion of piecewise continuity and establish a general theorem that provides insight into the representation of periodic functions using Fourier series.

Before stating the theorem, we need to introduce a new concept.

Definition. A function f defined on a closed interval $[a, b]$ is called *piecewise continuous* if the interval can be divided into a finite number of subintervals $[x_i, x_{i+1}]$:



so that f is continuous on each open subinterval (x_i, x_{i+1}). Additionally, all one-sided limits $\lim\limits_{x \to x_i^-} f(x)$ and $\lim\limits_{x \to x_i^+} f(x)$ exist and are finite. We will use the notation $c^- := \lim\limits_{x \to c^-} f(x)$ and $c^+ := \lim\limits_{x \to c^+} f(x)$ for such limits.

Theorem. Suppose a function f, which is $2L$-periodic, and its derivative f' are both piecewise continuous on the interval $-L \leq x < L$. Then, f admits a Fourier series representation:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right),$$

where the coefficients a_m and b_m are determined using previously derived formulas. This Fourier series converges to $f(x)$ at all points where f is continuous and to $\frac{f(x^+)+f(x^-)}{2}$ at points of discontinuity.

Lecture 12

Separation of Variables, Heat and Wave Equations

In the upcoming lectures, we will explore the fundamental partial differential equations governing heat conduction, wave propagation, and potential theory. These equations are intimately tied to diffusive processes, oscillatory phenomena, and steady-state behaviors, respectively, making them crucial in various branches of physics. From a mathematical standpoint, they hold significant importance.

The most well-developed and widely applicable equations are the linear second-order partial differential equations. Among them, the heat conduction equation, the wave equation, and the potential equation serve as prototypes for diffusive, oscillatory, and time-independent processes, respectively. Studying these foundational equations provides valuable insights into the broader realm of second-order linear partial differential equations.

Heat Conduction

Let's consider a heat conduction scenario involving a straight bar with uniform cross-sectional area and homogeneous material. We will align the x-axis along the axis of the bar, with $x = 0$ and $x = L$ representing the ends of the bar.

Assuming the sides of the bar are perfectly insulated, implying no heat transfer through them, and that the cross-sectional dimensions are sufficiently small for the temperature u to be considered constant on any given cross-section, we find that u becomes a function of only two variables: position (x) and time (t).

The temperature variation within the bar is governed by a partial differential equation known as the heat conduction equation, given by

$$u_t = \alpha^2 u_{xx}, \quad 0 < x < L, \quad t > 0,$$

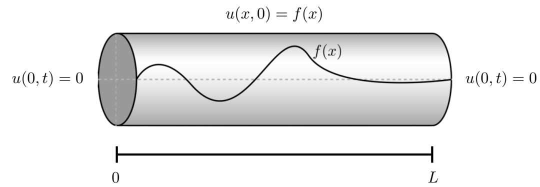
where α^2 represents a constant termed thermal diffusivity. This parameter α^2 relies solely on the material of the bar.

Additionally, we will assume that the initial temperature distribution within the bar is predetermined, represented as

$$u(x, 0) = f(x), \quad 0 \leq x \leq L,$$

where f is a given function. Furthermore, we consider the ends of the bar to be maintained at fixed temperatures: T_1 at $x = 0$ and T_2 at $x = L$. However, we only need to examine the scenario where $T_1 = T_2 = 0$. This special case allows us to simplify the more general problem. Therefore, moving forward, we will assume that u is always zero when $x = 0$ or $x = L$:

$$u(0,t) = 0, \quad u(L,t) = 0, \quad t \geq 0.$$



Let's look for a solution $u(x,t)$ in a simplified form, which is a product of two functions: one depending solely on x and the other solely on t. Thus, we have $u(x,t) = X(x)T(t)$. Substituting $u(x,t) = X(x)T(t)$ into the heat equation yields $\alpha^2 X_{xx} T = X T_t$, where primes denote derivatives with respect to the corresponding variable.

This equation can be rewritten as $\frac{X_{xx}}{X} = \frac{T_t}{\alpha^2 T}$. The key observation here is that the left-hand side is a function only of x, while the right-hand side is a function only of t. Hence, they can be equal as functions of their respective variables (for all values of x and t under consideration, where $0 < x < L$ and $t > 0$) only if both are constant, and the constant is the same. This observation leads us to set $\frac{X_{xx}}{X} = -\lambda$ and $\frac{T_t}{T} = -\alpha^2\lambda$ (we assume $\lambda > 0$ and introduce a minus sign in front of λ for convenience).

Solving $\frac{X_{xx}}{X} = -\lambda$ yields the differential equation $X_{xx} + \lambda X = 0$, with a general solution $X(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$. The boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$ imply $C_1 = 0$, and a nontrivial solution exists only for $\lambda_k = \frac{k^2\pi^2}{L^2}$ with $k \in \mathbb{Z}_{>0}$ (similar to the analogous example in one of the previous lectures). For such λ_k, the solution is given by $X_k = C_k \sin\left(\frac{k\pi x}{L}\right)$.

Next, we need to solve the equations $\frac{T_t}{T} = -\alpha^2\lambda_k$, which leads to $T'' + \frac{\alpha^2 k^2 \pi^2}{L^2} T = 0$.

The solution can be expressed as $T_k = \widehat{C} e^{-k^2\pi^2\alpha^2 t/L^2}$. Multiplying the solutions together, and neglecting constants of proportionality, we conclude that the functions $u_k(x,t) = X_k(x)T_k(x) = e^{-k^2\pi^2\alpha^2 t/L^2} \sin\left(\frac{k\pi x}{L}\right)$ satisfy the partial differential equation and the boundary conditions for each positive integer value of k. These functions u_k are sometimes called fundamental solutions of the heat conduction problem.

One more initial condition remains to be satisfied: $u(x,0) = f(x), 0 \leq x \leq L$. While, for a general choice of $f(x)$, the functions u_k will not satisfy this condition separately, their linear combination with appropriate coefficients will. Indeed, we need to recall that $u(0,0) = u(L,0) = 0$, implying $f(0) = f(L) = 0$. The problem of finding appropriate values of c_k's in the decomposition

$$f(x) = u(x,0) = \sum_{k\geq 1} c_k e^{-k^2\pi^2\alpha^2 t/L^2} \sin\left(\frac{k\pi x}{L}\right) = \sum_{k\geq 1} c_k \sin\left(\frac{k\pi x}{L}\right)$$

is nothing else but finding the Fourier series for $f(x)$ on the interval $[0,L]$. Hence, the coefficient c_k is given by the formula

$$c_k = \frac{2}{L}\int_0^L \sin\left(\frac{k\pi x}{L}\right) f(x)\,dx.$$

Remark. The 2 in the numerator comes from the fact that L is half the period. We can extend $f(x)$ by $f(x+L) = -f(L-x)$ for $0 < x \leq 2L$ to obtain a $2L$-periodic function. Since $f(0) = f(L) = 0$, the Fourier series for $f(x)$ cannot involve any cosines.

We conclude that a solution to the heat equation

$$\begin{cases} u_t = \alpha^2 u_{xx} \\ u(x,0) = f(x) \\ u(0,t) = u(L,t) = 0, \quad t \geq 0 \end{cases}$$

is given by

$$y = \sum_{k\geq 1} \left(\frac{2}{L} \int_0^L \sin\left(\frac{k\pi x}{L}\right) f(x)\, dx \right) e^{-k^2\pi^2\alpha^2 t/L^2} \sin\left(\frac{k\pi x}{L}\right).$$

Example. Find the solution of the heat conduction problem

$$\begin{cases} u_t = 0.01 u_{xx}, & 0 < x < 1, \quad t \geq 0; \\ u(0,t) = 0, & u(1,t) = 0, \quad t \geq 0; \\ u(x,0) = \sin(2\pi x) - \sin(5\pi x), & 0 \leq x \leq 1. \end{cases}$$

We start by noting that $L = 1$ and $\alpha^2 = 0.01$. The solution is then expressed as

$$y = \sum_{k\geq 1} \left(2 \int_0^1 \sin(k\pi x) f(x)\, dx \right) e^{-k^2\pi^2 0.01 t} \sin(k\pi x).$$

which, at $t = 0$, simplifies to $\sum\limits_{k\geq 1} c_k \sin(k\pi x)$. It should match $\sin(2\pi x) - \sin(5\pi x)$. Hence, we deduce $c_2 = c_5 = 1$ and $c_i = 0$ for $i \notin \{2, 5\}$. Substituting these values back into the equation, we derive the solution:

$$u(x,t) = e^{-0.04\pi^2 t} \sin(2\pi x) - e^{-0.25\pi^2 t} \sin(5\pi x).$$

Wave Equation

Another essential partial differential equation encountered frequently in applied mathematics is the wave equation. This equation arises in the mathematical analysis of phenomena involving the propagation of waves in a continuous medium. It finds applications in various studies, including acoustic waves, water waves, electromagnetic waves, and seismic waves. The elastic string can be conceptualized as a violin string, a guy wire, or even an electric power line.

Suppose the string is set in motion (by plucking, for example) so that it vibrates in a vertical plane. Let $u(x,t)$ denote the vertical displacement experienced by the string at the point x at time t. Neglecting damping effects such as air resistance and assuming the amplitude of the motion is not too large, $u(x,t)$ satisfies the partial differential equation:

$$a^2 u_{xx} = u_{tt}$$

in the domain $0 < x < L, t > 0$. This equation is known as the one-dimensional wave equation. Here, the constant a represents the velocity of propagation of waves along the string. To describe the motion of the string fully, it's necessary to specify suitable initial and boundary conditions for the displacement $u(x,t)$. The ends of the string are assumed to remain fixed, leading to the boundary conditions:

$$u(0,t) = 0, \quad u(L,t) = 0, \quad t \geq 0$$

Since the differential equation is of second order with respect to t, it's reasonable to prescribe two initial conditions. These are the initial position of the string:

$$u(x,0) = f(x), \quad 0 \leq x \leq L$$

and its initial velocity:

$$u_t(x,0) = g(x), \quad 0 \leq x \leq L,$$

where f and g are given functions. For consistency, it's also necessary to impose that:

$$f(0) = f(L) = 0, \quad g(0) = g(L) = 0.$$

Elastic Strings with Nonzero Initial Displacement

Elastic strings, when disturbed from their equilibrium position and then released at time $t = 0$ with zero velocity, exhibit behavior. which is mathematically described by the wave equation

$$a^2 u_{xx} = u_{tt}, \quad 0 < x < L, \quad t > 0,$$

with boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0, \quad t \geq 0,$$

and initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = 0, \quad 0 \leq x \leq L,$$

where f is a given function representing the initial displacement of the string. Assuming $u(x,t) = X(x)T(t)$ and substituting into the equation, we obtain

$$\frac{X_{xx}}{X} = \frac{T_{tt}}{a^2 T} = -\lambda,$$

where λ is a separation constant. Thus, $X(x)$ and $T(t)$ satisfy the ordinary differential equations

$$X_{xx} + \lambda X = 0, \quad T_{tt} + a^2 \lambda T = 0.$$

Furthermore, the boundary conditions translate into the following conditions on $X(x)$ and $T(t)$:

$$X(0) = 0, \quad X(L) = 0, \quad T'(0) = 0.$$

Solving the former equation, we find nontrivial solutions only for $\lambda_k = \frac{k^2\pi^2}{L^2}$ with $k \in \mathbb{Z}_{>0}$, given by $X_k = C_k \sin\left(\frac{k\pi x}{L}\right)$. Substituting into the second equation yields

$$T_{tt} + a^2 \frac{k^2\pi^2}{L^2} T = 0,$$

with general solution

$$T = C_1 \cos\left(\frac{k\pi a t}{L}\right) + C_2 \sin\left(\frac{k\pi a t}{L}\right).$$

The initial condition $T'(0) = C_2 \frac{k\pi a}{L} = 0$ implies $C_2 = 0$, resulting in $T = C_1 \cos\left(\frac{k\pi a t}{L}\right)$. Consequently, the functions $C_k \sin\left(\frac{k\pi x}{L}\right) \cos\left(\frac{k\pi a t}{L}\right)$ with $k \in \mathbb{Z}_{>0}$ are solutions to the partial differential equation $a^2 u_{xx} = u_{tt}$ and satisfy the initial conditions $X(0) = 0$, $X(L) = 0$, $T'(0) = 0$. As before, we assume that $u(x,t)$ has the form

$$u(x,t) = \sum_{k\geq 1} c_k u_k(x,t) = \sum_{k \geq 1} c_k \sin\left(\frac{k\pi x}{L}\right) \cos\left(\frac{k\pi a t}{L}\right),$$

where the constants c_k are the coefficients in the Fourier series decomposition of $u(x,0) = f(x)$.

General Case

Let's refine our previous scenario by considering the string set in motion from its equilibrium position with a specified velocity instead of position. This adjustment leads to the vertical displacement $u(x,t)$ satisfying the wave equation

$$a^2 u_{xx} = u_{tt}, \quad 0 < x < L, \quad t > 0;$$

subject to boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0, \quad t \geq 0;$$

and the initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = g(x), \quad 0 \le x \le L,$$

where $g(x)$ denotes the initial velocity at point x of the string. This time, the associated initial condition is $T(0) = 0$ instead of $T'(0) = 0$. We can see that the functions $C_k \sin\left(\frac{k\pi x}{L}\right)\sin\left(\frac{k\pi at}{L}\right)$ with $k \in \mathbb{Z}_{>0}$ satisfy the PDE and all conditions except for $u_t(x,0) = g(x)$. We seek a solution in the form

$$v(x,t) = \sum_{k\ge 1} c_k u_k(x,t) = \sum_{k\ge 1} c_k \sin\left(\frac{k\pi x}{L}\right)\sin\left(\frac{k\pi at}{L}\right),$$

and determine the constants c_k from the coefficients in the Fourier series decomposition of the function $u_t(x,0) = g(x)$. Finally, the solution in the case of general initial conditions: $u(x,0) = f(x)$, $u_t(x,0) = g(x)$, $0 \le x \le L$, can be obtained as the superposition sum of the two solutions $u(x,t) + v(x,t)$.

Lecture 13
Laplace Equation

In today's lecture, we will explore the fundamental concepts surrounding the Laplace equation, a cornerstone of mathematical physics and engineering. This equation, named after the French mathematician Pierre-Simon Laplace, arises in various fields, including electromagnetism, fluid dynamics, and heat conduction.

For instance, consider the temperature $u = u(x,y,t)$ within a two-dimensional plate, governed by the two-dimensional heat equation:

$$u_t = a^2(u_{xx} + u_{yy})$$

Here, (x,y) varies across the plate's interior, and $t > 0$ denotes time. Solving this equation necessitates specifying the initial temperature $u(x,y,0)$ and boundary conditions. However, as time progresses ($t \to \infty$), the temperature stabilizes so

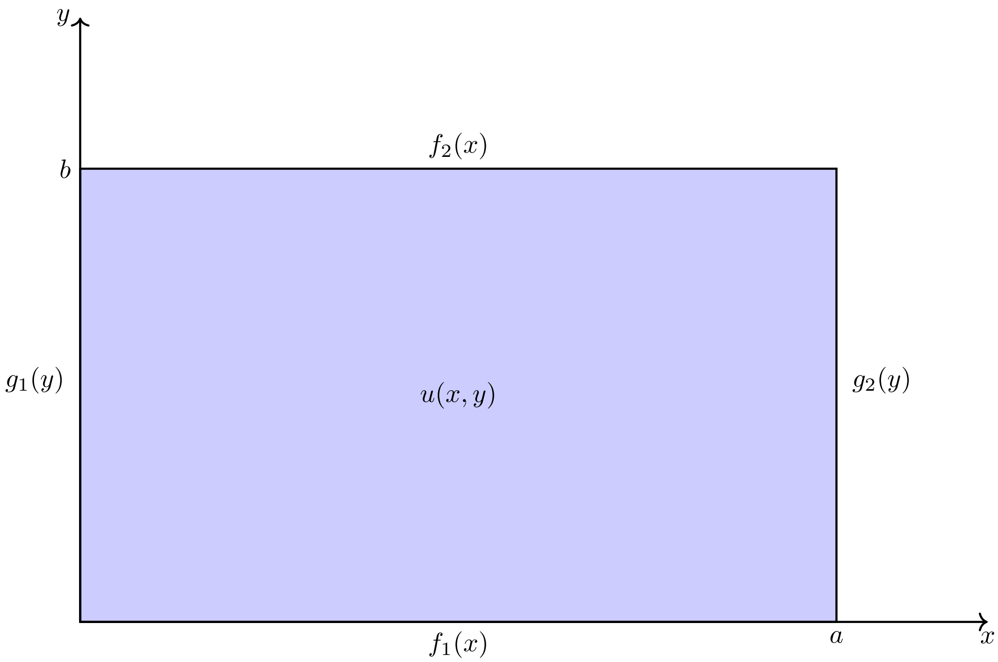
$$\lim_{t\to\infty} u_t(x,y,t) = 0.$$

The temperature approaches a steady-state distribution $u = u(x,y)$, satisfying the Laplace equation:

$$u_{xx} + u_{yy} = 0.$$

This is a (two dimensional) Laplace's equation. We seek solutions of the in a region R that satisfy specified conditions – called boundary conditions – on the boundary of R. Solving boundary value problems over general regions can be difficult, so we consider only very simple, rectangular regions. The possible boundary conditions for this region can be written as

$$\begin{cases} (1-\alpha)u(x,0) + \alpha u_y(x,0) = f_1(x) \\ (1-\beta)u(x,b) + \beta u_y(x,b) = f_2(x) \\ (1-\gamma)u(0,y) + \gamma u_x(0,y) = g_1(y) \\ (1-\delta)u(a,y) + \delta u_x(a,y) = g_2(y) \end{cases}$$



where α, β, γ, and δ can each be either 0 or 1; thus, there are 16 possibilities. This is a **Dirichlet problem** if

$$\alpha = \beta = \gamma = \delta = 0$$

or a **Neumann problem** if

$$\alpha = \beta = \gamma = \delta = 1.$$

The problems corresponding to other quadruples of values $(\alpha, \beta, \gamma, \delta)$ are called **mixed**.

Remark. Notice that to solve the Laplace equation $u_{xx} + u_{yy} = 0$ with parameters $(\alpha, \beta, \gamma, \delta)$ subject to boundary conditions $(f_1(x), f_2(x), g_1(y), g_2(y))$, it suffices to find solutions $u_{f_1}, u_{f_1}, u_{g_1}, u_{g_2}$ corresponding to boundary conditions $(f_1(x), 0, 0, 0), (0, f_2(x), 0, 0), (0, 0, g_1(x), 0), (0, 0, 0, g_2(x))$ respectively, and then take their superposition $u(x, y) = u_{f_1} + u_{f_1} + u_{g_1} + u_{g_2}$. Therefore, it suffices to focus on problems where only one of the functions f_0, f_1, g_0, g_2 is not identically zero. Each such equation has homogeneous boundary conditions on three sides of the rectangle and a nonhomogeneous boundary condition on the fourth.

Dirichlet Problem for a Rectangle

Consider the problem of finding the function u satisfying Laplace's equation

$$u_{xx} + u_{yy} = 0,$$

in the rectangle $0 < x < a$, $0 < y < b$, and also satisfying the boundary conditions

$$u(x, 0) = 0, \quad u(x, b) = 0, \quad 0 < x < a,$$

$$u(0, y) = 0, \quad u(a, y) = f(y), \quad 0 \le y \le b,$$

where f is a given function on $0 \le y \le b$. As before, we assume that

$$u(x, y) = X(x)Y(y)$$

and substitute for u in the initial equation, which transforms into a system of two second-order ODEs:

$$\frac{X_{xx}}{X} = \lambda, \quad -\frac{Y_{yy}}{Y} = \lambda,$$

where λ is the separation constant. Our boundary conditions imply that

$$X(0) = 0 \quad \text{and} \quad Y(0) = 0, \quad Y(b) = 0.$$

The equation

$$\frac{Y_{yy}}{Y} = -\lambda \Leftrightarrow Y_{yy} + \lambda Y = 0$$

has a general solution given by $Y(y) = C_1 \cos(\sqrt{\lambda} y) + C_2 \sin(\sqrt{\lambda} y)$. Applying the boundary condition $Y(0) = 0$ leads to $C_1 = 0$. Moreover, a nontrivial solution satisfying the second boundary condition $Y(b) = 0$ exists only for $\lambda_k = \frac{k^2\pi^2}{L^2}$ for $k \in \mathbb{Z}_{\geq 1}$, which simplifies to $\lambda = \frac{k^2\pi^2}{b^2}$, yielding $Y_k(y) = C_k \sin\left(\frac{k\pi y}{b}\right)$.

Next, we solve the first equation:

$$\frac{X_{xx}}{X} = \frac{k^2\pi^2}{b^2} \Leftrightarrow X_{xx} - \frac{k^2\pi^2}{b^2} X = 0$$

subject to the boundary condition $X(0) = 0$. The general solution is $X_k = C_1 e^{k\pi x/b} + C_2 e^{-k\pi x/b}$. Recalling the hyperbolic functions $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and $\cosh(x) = \frac{e^x + e^{-x}}{2}$, we can pick these functions as a fundamental set and write

$$X_k = C_1 \cosh\left(\frac{k\pi x}{b}\right) + C_2 \sinh\left(\frac{k\pi x}{b}\right).$$

As $X_k(0) = C_1 \cosh(0) + C_2 \sinh(0) = C_1 = 0$, we obtain the solutions $X_k = \widehat{C}_k \sinh\left(\frac{k\pi x}{b}\right)$. We conclude that the functions

$$u_k(x, y) = \sinh\left(\frac{k\pi x}{b}\right) \sin\left(\frac{k\pi y}{b}\right)$$

satisfy the differential equation and all the homogeneous boundary conditions for each value of $k \in \mathbb{Z}_{\geq 1}$. To satisfy the remaining nonhomogeneous boundary condition at $x = a$, we assume, as before, that we can represent the solution $u(x, y)$ as a superposition

$$u(x, y) = \sum_{k \geq 1} c_k u_k = \sum_{k \geq 1} c_k \sinh\left(\frac{k\pi x}{b}\right) \sin\left(\frac{k\pi y}{b}\right)$$

The coefficients c_k are determined by the boundary condition

$$u(a, y) = \sum_{k \geq 1} c_k \sinh\left(\frac{k\pi a}{b}\right) \sin\left(\frac{k\pi y}{b}\right) = \widetilde{f}(y),$$

which represents the Fourier series of $\widetilde{f}$. Here, $\widetilde{f}$ is obtained by first extending $f(y)$ to an odd function defined as

$$\widetilde{f}(y) = \begin{cases} f(y), & 0 \leq y \leq b \\ -f(y), & -b \leq y < 0 \end{cases}$$

and then periodically extending it to an odd function with period $2b$ ($\widetilde{f}(y) = f(y + 2b)$). Therefore,

$$c_k \sinh\left(\frac{k\pi a}{b}\right) = \frac{1}{b} \int_{-b}^{b} f(y) \sin\left(\frac{k\pi y}{b}\right) \Leftrightarrow c_k = \frac{\int_{-b}^{b} \widetilde{f}(y) \sin\left(\frac{k\pi y}{b}\right)}{b \sinh\left(\frac{k\pi a}{b}\right)}$$

and the solution is

$$u(x, y) = u(x, y) = \frac{1}{b} \sum_{k \geq 1} \frac{\int_{-b}^{b} f(y) \sin\left(\frac{k\pi y}{b}\right)}{\sinh\left(\frac{k\pi a}{b}\right)} \sinh\left(\frac{k\pi x}{b}\right) \sin\left(\frac{k\pi y}{b}\right)$$

Notice that for large values of x, the term e^{-x} becomes a small positive number. Consequently, the hyperbolic sine function, $\sinh(x) = \frac{e^x + e^{-x}}{2}$, can be approximated by $\frac{e^x}{2}$. For $k \gg 0$, this approximation yields:

$$\frac{\sinh\left(\frac{k\pi x}{b}\right)}{\sinh\left(\frac{k\pi a}{b}\right)} \approx \frac{e^{k\pi x/b}}{e^{k\pi a/b}} = e^{k\pi(x-a)/b}.$$

Recall that $0 < x < a$, which implies $x - a < 0$. Therefore, as k tends to infinity, the term $e^{k\pi(x-a)/b}$ tends to 0.

Moreover, since $\left|\sin\left(\frac{k\pi y}{b}\right)\right| \leq 1$ and $\left|\int\limits_{-b}^{b} \widetilde{f}(y)\sin\left(\frac{k\pi y}{b}\right)\right| \leq 2b \max\limits_{y\in[0,b]} |f(y)|$, the series converges rapidly for x sufficiently far from a.

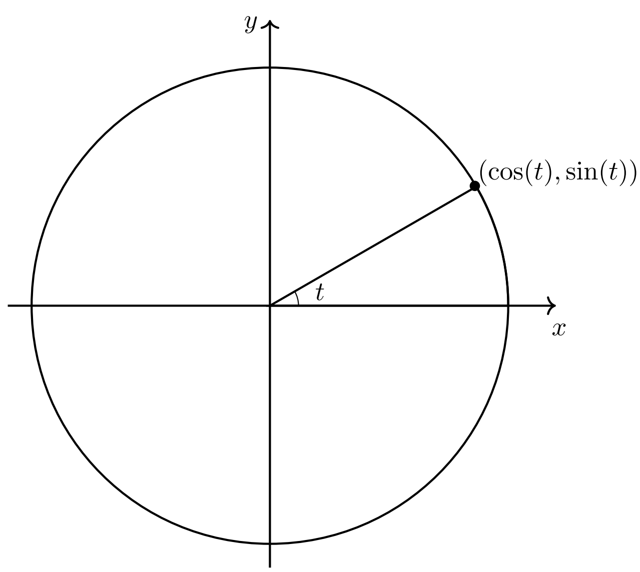
Optional: Hyperbolic Functions

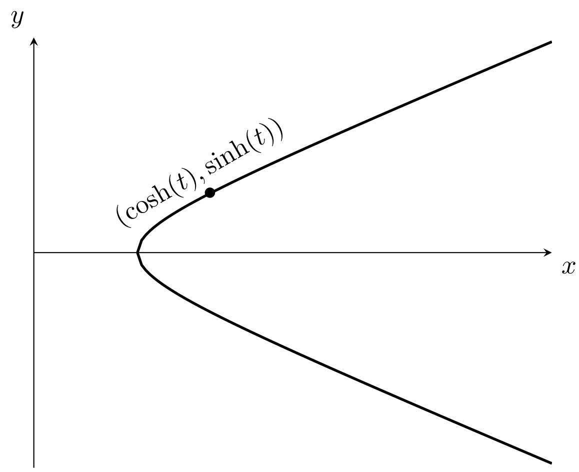
Consider the parameterization of the unit circle given by $(x, y) = (\cos(t), \sin(t))$ for $t \in [0, 2\pi)$.

Notice that

$$\cosh^2(t) - \sinh^2(t) = \left(\frac{e^t + e^{-t}}{2}\right)^2 - \left(\frac{e^t - e^{-t}}{2}\right)^2 = \frac{e^{2t} + 2e^t e^{-t} + e^{-2t}}{4} - \frac{e^{2t} - 2e^t e^{-t} + e^{-2t}}{4} = \frac{4}{4} = 1.$$

By replacing cosine and sine with their hyperbolic counterparts, we can parameterize the right branch of the hyperbola $x^2 - y^2 = 1$. This parameterization is given $(x, y) = (\cosh(t), \sinh(t))$, which exhibits the hyperbolic nature of these functions.





y
$(\cosh(t), \sinh(t))$
x