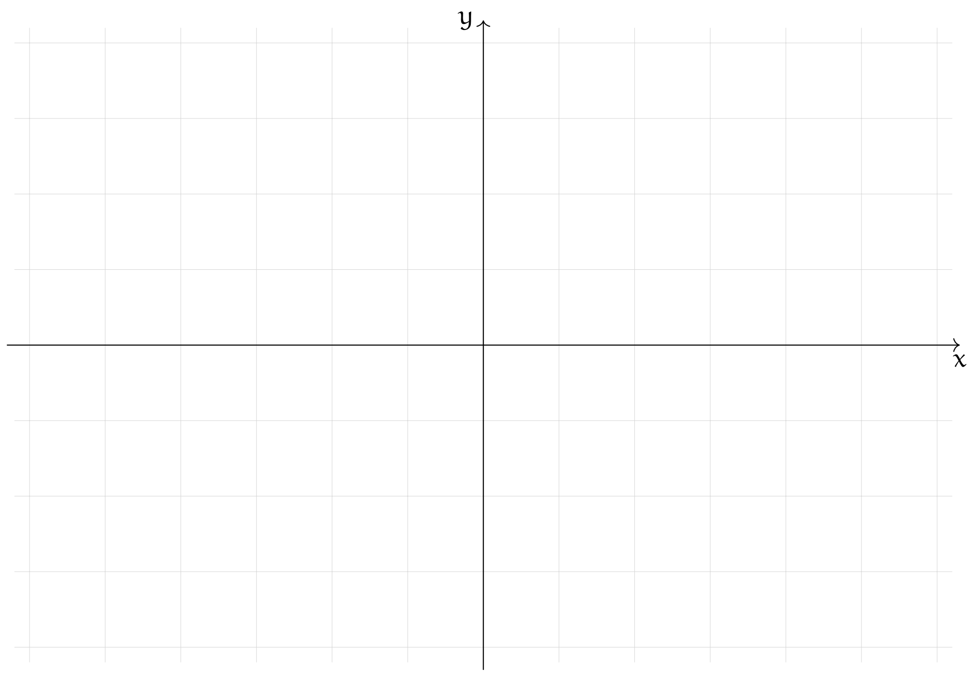
MATH 146B: Ordinary and Partial Differential Equations

Homework 3

Fourier series

Problem 1. Consider the function $g(x) = \begin{cases} -2, & \text{if } -2 \leq x < -1, \\ 1, & \text{if } -1 \leq x < 0, \\ -1, & \text{if } 0 \leq x < 1, \\ 2, & \text{if } 1 \leq x < 2. \end{cases}$ $g(x+4) = g(x)$.

(a) (5 points) Sketch the graph of $g(x)$ on the interval $[-6, 6]$.



(b) (5 points) Analyze the function $g(x)$ for any symmetries, such as whether it is even or odd (considering potential vertical shifts). Based on these symmetries, what insights can be drawn regarding certain Fourier coefficients?

(c) (5 points) Determine the Fourier series **coefficients** for $g(x)$. You do not need to explicitly write out the series.

(d) (5 points) Using the Fourier series representation of $g(x)$ as provided:

$$g(x) = \frac{2}{\pi} \sum_{m \geq 0} \frac{1}{2m+1} \sin\left(\frac{(2m+1)\pi x}{2}\right) - \frac{12}{\pi} \sum_{m \geq 0} \frac{1}{2m+2} \sin\left(\frac{(2m+2)\pi x}{2}\right),$$

verify its convergence at $x = 0$ to the number prescribed by the Theorem on page 5 of Lecture 11 notes.

Hint: utilize your graph sketch from part (a).

Problem 2. Consider the function $f(x) = x^2$ on the interval $[-\pi, \pi]$.

(a) (5 points) Analyze the function $f(x)$ for any symmetries, such as whether it is even or odd (considering potential vertical shifts). Based on these symmetries, what insights can be drawn regarding certain Fourier coefficients?

(b) (10 points) Find Fourier series for $f(x)$.

Extra-credit. (5 points) Use your answer in the previous problem to establish the identity $\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Hint: Evaluate the series by substituting $x = \pi$. According to the theorem discussed in Lecture 11, it should converge to $f(\pi) = \pi^2$.