MATH 146B: Ordinary and Partial Differential Equations

Final Bonus

Solutions

An example of non-analytic smooth function

Problem 1. Consider the function

$$f(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \le 0. \end{cases}$$

(a) (2 points) Show that the function $f(x)$ is continuous at $x = 0$.

Hint: compare the one-sided limits $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$.

Solution: to show that the function $f(x)$ is continuous at $x = 0$, we need to check if the one-sided limits $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$ are equal.

For $x > 0$, we have $f(x) = e^{-1/x}$, so $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{-1/x} = 0$.

For $x \le 0$, we have $f(x) = 0$, so $\lim_{x\to 0^-} f(x) = 0$.

Since $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 0$, the function $f(x)$ is continuous at $x = 0$.

(b) (3 points) Show that $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x)$ and compute the value of $f'(0)$.

Solution: let's compute the derivative $f'(x)$ for $x > 0$:

$$f'(x) = e^{-1/x} \cdot (-1/x)' = e^{-1/x} \cdot \frac{1}{x^2}.$$

Thus, $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} \frac{e^{-1/x}}{x^2} = \lim_{t\to\infty} t^2 e^{-t} = 0 = \lim_{x\to 0^-} f'(x)$ (here $t = \frac{1}{x}$).

Remark. Similarly, it can be shown that $\lim_{x\to 0^-} f^{(k)}(x) = \lim_{x\to 0^+} f^{(k)}(x) = 0$ for any positive integer $k > 0$. Consequently, $f^{(k)}(0) = 0$ for all positive integers $k > 0$.

(c) (2 points) Use your findings from (a) and (b) along with the remark to construct the Taylor series expansion for $f(x)$ centered at 0. Explain why this series does not converge to the actual value of $f(x)$ for all x in any interval about 0.

Solution:

Separation of variables

Problem 2. Consider the partial differential equation $tu_{xx} + 5u_t = 0$.

(a) (3 points) Look for a solution in the form $u(x,t) = X(x)T(t)$ and transform the equation into a system of two ODEs depending on separation constant λ.

Solution: to solve the partial differential equation $tu_{xx} + 5u_t = 0$, we assume a solution of the form $u(x,t) = X(x)T(t)$. Substituting this into the equation, we get:

$$tX''(x)T(t) + 5X(x)T'(t) = 0 \Leftrightarrow \frac{tX''(x)}{X(x)} + \frac{5T'(t)}{T(t)} = 0$$

Rearranging terms, we obtain:

$$\frac{X''(x)}{X(x)} = -\frac{5T'(t)}{tT(t)}$$

To satisfy this equation for all x and t, both sides must be equal to a constant, which we denote as $-\lambda$. So, we have:

$$\frac{X''(x)}{X(x)} = -\lambda \quad \text{and} \quad \frac{5T'(t)}{tT(t)} = \lambda.$$

The first equation gives us a second-order ordinary differential equation (ODE) for $X(x)$, and the second equation gives us a first-order ODE for $T(t)$. We have separated the original partial differential equation into a system of two ordinary differential equations depending on the separation constant λ.

(b) (4 points) Solve the equations that you obtained in (a) and give the family of solutions (depending on $\lambda > 0$) of the initial equation produced this way.

Solution: to solve the equations obtained in part (a), let's start with the equation for $X(x)$:

$$X''(x) + \lambda X(x) = 0.$$

The general solution is of the form $X(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$.

Next, let's solve the equation for $T(t)$:

$$T'(t) = 0.2\lambda t T(t)$$

This is a separable first-order ODE, and its solution is given by $T(t) = Ce^{0.1\lambda t^2}$.

Combining the solutions for $X(x)$ and $T(t)$, we obtain the family of solutions for the initial equation:

$$u(x,t) = e^{0.1\lambda t^2}(C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)).$$

Heat equation on $\mathbb{R}$ and $\mathbb{R}^2$

Problem 3.

(a) (3 points) Check that the function $u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$ satisfies the heat equation $u_t = u_{xx}$ for $x \in \mathbb{R}$ and $t > 0$.

Solution: to check if $u(x,t)$ satisfies the heat equation, we need to compute and compare the necessary partial derivatives with respect to t and x, and then verify if they satisfy the equation $u_t = u_{xx}$:

$$u_t = \frac{\partial}{\partial t}\left(\frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}\right) = \frac{-e^{-\frac{x^2}{4t}}}{\sqrt{4\pi} \cdot 2t^{3/2}} + \frac{x^2 e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t} \cdot 4t^2} = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}\left(-\frac{1}{2t} + \frac{x^2}{4t^2}\right)$$
$$u_{xx} = \frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}\right) = \frac{1}{\sqrt{4\pi t}}\frac{\partial^2}{\partial x^2}\left(e^{-\frac{x^2}{4t}}\right) = \frac{1}{\sqrt{4\pi t}}\frac{\partial}{\partial x}\left(\frac{-2xe^{-\frac{x^2}{4t}}}{4t}\right) = \frac{1}{\sqrt{4\pi t}}\left(\frac{-2e^{-\frac{x^2}{4t}}}{4t} + \frac{-4x^2 e^{-\frac{x^2}{4t}}}{16t^2}\right)$$
$$= \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}\left(-\frac{1}{2t} + \frac{x^2}{4t^2}\right),$$

which are equal, confirming that $u(x,t)$ satisfies the heat equation $u_t = u_{xx}$.

(b) (3 points) Check that the function $u(x,y,t) = u(x,t)u(y,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \cdot \frac{1}{\sqrt{4\pi t}} e^{-\frac{y^2}{4t}} = \frac{1}{4\pi t} e^{\frac{-x^2-y^2}{4t}}$ satisfies the heat equation $u_t = u_{xx} + u_{yy}$ for $(x,y) \in \mathbb{R}^2$ and $t > 0$.

Hint: Use the product rule and the result from part (a).

Solution: to check if $u(x,y,t)$ satisfies the heat equation, we can first express $u(x,y,t)$ as a product of two functions $u(x,t)$ and $u(y,t)$, both of which are solutions to the respective one-dimensional heat equation as shown in part (a). Then we can apply the product rule and verify if the resulting expression satisfies the equation $u_t = u_{xx} + u_{yy}$.

$$u_{xx} = u_{xx}(x,t)u(y,t),\; u_{yy} = u(x,t)u_{yy}(y,t)$$
$$u_t = u_t(x,t)u(y,t) + u(x,t)u_t(y,t) = u_{xx}(x,t)u(y,t) + u(x,t)u_{yy}(y,t) \qquad \checkmark$$