MATH 146B: Ordinary and Partial Differential Equations

Final Bonus

An example of non-analytic smooth function

Problem 1. Consider the function
$$f(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \leq 0. \end{cases}$$

(a) (2 points) Show that the function $f(x)$ is continuous at $x = 0$.

Hint: compare the one-sided limits $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$.

(b) (3 points) Show that $\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^+} f'(x)$ and compute the value of $f'(0)$.

Remark. Similarly, it can be shown that $\lim_{x \to 0^-} f^{(k)}(x) = \lim_{x \to 0^+} f^{(k)}(x) = 0$ for any positive integer $k > 0$. Consequently, $f^{(k)}(0) = 0$ for all positive integers $k > 0$.

(c) (2 points) Use your findings from (a) and (b) along with the remark to construct the Taylor series expansion for $f(x)$ centered at 0. Explain why this series does not converge to the actual value of $f(x)$ for all x in any interval about 0.

Separation of variables

Problem 2. Consider the partial differential equation $tu_{xx} + 5u_t = 0$.

(a) (3 points) Look for a solution in the form $u(x, t) = X(x)T(t)$ and transform the equation into a system of two ODEs depending on separation constant λ.

(b) (4 points) Solve the equations that you obtained in (a) and give the family of solutions (depending on $\lambda > 0$) of the initial equation produced this way.

Heat equation on $\mathbb{R}$ and $\mathbb{R}^2$

Problem 3.

(a) (3 points) Check that the function $u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$ satisfies the heat equation $u_t = u_{xx}$ for $x \in \mathbb{R}$ and $t > 0$.

(b) (3 points) Check that the function $u(x, y, t) = u(x, t)u(y, t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \cdot \frac{1}{\sqrt{4\pi t}} e^{-\frac{y^2}{4t}} = \frac{1}{4\pi t} e^{\frac{-x^2-y^2}{4t}}$ satisfies the heat equation $u_t = u_{xx} + u_{yy}$ for $(x, y) \in \mathbb{R}^2$ and $t > 0$.

Hint: use the product rule and the result from part (a).

Remark. Note that you may have recognized that $u(x, t)$ is the density function for the normal (Gaussian) distribution from probability theory. This connection is further emphasized by the fact that Gaussian distributions are closely related to Brownian motion, a stochastic process that models the random movement of particles suspended in a fluid. In particular, the density function $u(x, t)$ describes the probability distribution of the position of a particle undergoing Brownian motion at time t, starting from an initial position of $x = 0$. This connection highlights the deep interplay between probability theory and the study of diffusion processes, such as heat conduction, which are fundamental in various scientific disciplines.