Spring 2024

MATH 11: Introduction to Discrete Structures

Homework 5

Problem 1. (20 points) Consider the sets $A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ and $B = \{b_1, b_2\}$.

(a) (5 points) Calculate the number of one-to-one (injective) functions from set A to set B.

(b) (5 points) Calculate the number of one-to-one (injective) functions from set B to set A.

(c) (5 points) Calculate the number of onto (surjective) functions from set A to set B.

(d) (5 points) Calculate the number of onto (surjective) functions from set B to set A.

Problem 2. (20 points) (It may be helpful to refer to the examples on pages 3–5 of the Lecture 14 notes.) The sequence $\{a_n\}_{n\in\mathbb{Z}_{\geq 0}}$ is defined recursively by $a_0 = 4$, $a_1 = 3$, and $a_n = -5a_{n-1} + 14a_{n-2}$ for $n \geq 2$.

(a) (5 points) Write down the characteristic polynomial $\chi(t)$ for the given sequence and find its roots.

(b) (5 points) Express the general term a_n in the form given in the Proposition on page 2 of Lecture 14 notes (including parameters c_1 and c_2).

(c) (5 points) Determine the values of c_1 and c_2 using the initial terms a_0 and a_1, and obtain an expression for the general term a_n in terms of the roots of $\chi(t)$.

(d) (5 points) Compute the value of a_2 using your formula from part (c) and compare it with the value obtained from the original recursive definition.

Problem 3. (20 points) Consider the sequence $\{a_n\}$ defined recursively by $a_n = 2a_{n-1} - 1$ for $n \geq 1$ with $a_0 = 5$.

(a) (5 points) Compute the values of the first 7 terms of the sequence (a_0 through a_6).

(b) (5 points) Encode your answers into a generating function $S(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + \dots$ and compute the first 7 terms of the expression $S(t) - 2tS(t) + \frac{1}{1-t}$. Specifically, fill in the blanks in

$$S(t) - 2tS(t) + \frac{1}{1-t} = \_ + \_t + \_t^2 + \_t^3 + \_t^4 + \_t^5 + \_t^6 + \dots$$

Hint: Recall that $\frac{1}{1-t} = 1 + t + t^2 + \dots$

(c) (5 points) Verify the equality $S(t) - 2tS(t) + \frac{1}{1-t} = 6$ and use it to express $S(t)$ as a rational function of t.

(d) (5 points) Enter your answer from part (c) into the series expansion tool with the following settings:

- **Variable**: t
- **Expansion around the value**: 0
- **Precision (Until Order** $N = 6$**)**

Confirm that the first seven coefficients of the series expansion match the values you found in part (a). If they do, write "yes" to claim the points.

Problem 4. (5 points) On page 2 of Lecture 14, it is explained that a homogeneous linear recurrence relation of order 2 takes the form:

$$a_n = \beta a_{n-1} + \theta a_{n-2}$$

where β and θ are constants. The corresponding generating function $A(t) = \sum_{n \geq 0} a_n t^n$ can be expressed as:

$$A(t) = \frac{a_0 + (a_1 - \beta a_0)t}{1 - \beta t - \theta t^2}.$$

Verify for yourself that this expression for $A(t)$ is correct. If you are convinced, write "yes" to claim the points.