MATH 11: Introduction to Discrete Structures

# Final Review

**Problem 1.** (2 points) If you roll two fair six-sided dice, what is the probability of getting

(a) a sum between 7 and 9 (inclusive)?

**Solution.** $P(7 \le \text{sum} \le 9) = P(\text{sum} = 7) + P(\text{sum} = 8) + P(\text{sum} = 9) = \frac{6+5+4}{36} = \frac{15}{36} = \frac{5}{12}$:

| $n_1 / n_2$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 |

(b) a sum strictly less than 7 or strictly greater than 9?

**Solution.**$P(\text{sum} < 7 \cup \text{sum} > 9) = 1 - P(7 \le \text{sum} \le 9) = 1 - \frac{5}{12} = \frac{7}{12}$.

**Problem 2.** (3 points) Suppose we have a random experiment with sample space $S$ and two events $A$ and $B$. Determine whether the following statements are true or false, and justify your answer:

(a) For any two events $A$ and $B$, we have $P(A \cap B) = P(A)P(B)$.

**Solution.** False. This is only true if $A$ and $B$ are independent events.

(b) If events $A$ and $B$ are mutually exclusive, then $P(A \cap B) = 0$.

**Solution.** True. Mutually exclusive events cannot occur simultaneously, so their intersection is empty.

(c) If events $A$ and $B$ are independent, then $P(A|B) = P(A)$.

**Solution.** True. If $A$ and $B$ are independent, the occurrence of $B$ does not affect the probability of $A$.

**Problem 3.** (3 points) Let $X$ be a discrete random variable with the following probability mass function:

$$P(X = k) = \begin{cases} 0.2 & \text{if } k = 1 \\ 0.3 & \text{if } k = 2 \\ 0.5 & \text{if } k = 3 \\ 0 & \text{otherwise} \end{cases}$$

Calculate the following probabilities.

(a) $P(X = 2)$

**Solution.** $P(X = 2) = 0.3$.

(b) $P(X < 3)$

**Solution.** $P(X < 3) = P(X = 1) + P(X = 2) = 0.2 + 0.3 = 0.5$.

(c) $P(X \geq 2)$

**Solution.** $P(X \geq 2) = P(X = 2) + P(X = 3) = 0.3 + 0.5 = 0.8$.

**Problem 4.** (1 point) Consider two events $A$ and $B$ such that $P(A) = 0.3$ and $P(B) = 0.4$. If $A$ and $B$ are independent, what is $P(A \cap B)$?

**Solution.** As the events are independent, $P(A \cap B) = P(A)P(B) = 0.3 \cdot 0.4 = 0.12$.

**Problem 5.** (1 point) Consider two events $E$ and $F$ such that $P(E) = 0.6$ and $P(F) = 0.7$. If $E$ and $F$ are independent, what is $P(E \cup F)$?

**Solution.** $P(E \cup F) = P(E) + P(F) - P(E)P(F) = 0.6 + 0.7 - 0.6 \cdot 0.7 = 1.3 - 0.42 = 0.88$.

**Problem 6.** (1 point) Let $X$ be a random variable representing the number of heads obtained when flipping a coin 3 times. Find the probability mass function of $X$.

**Solution.** When flipping a fair coin three times, the possible values of $X$ are $0, 1, 2$, and $3$, representing the number of heads obtained. The number of heads in 3 coin flips follows a binomial distribution with parameters $n = 3$ and $p = 0.5$. The probability mass function (PMF) of a binomial random variable $X$ is given by:

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

Calculating this for $k = 0, 1, 2$, and $3$:

$$\begin{aligned}
P(X = 0) &= \binom{3}{0} (0.5)^3 = 1 \cdot \left(\frac{1}{8}\right) = \frac{1}{8}; \\
P(X = 1) &= \binom{3}{1} (0.5)^3 = 3 \cdot \left(\frac{1}{8}\right) = \frac{3}{8}; \\
P(X = 2) &= \binom{3}{2} (0.5)^3 = 3 \cdot \left(\frac{1}{8}\right) = \frac{3}{8}; \\
P(X = 3) &= \binom{3}{3} (0.5)^3 = 1 \cdot \left(\frac{1}{8}\right) = \frac{1}{8}
\end{aligned}$$

Therefore, the PMF is

$$P(X = k) = \begin{cases} \frac{1}{8} & \text{if } k = 0, \\ \frac{3}{8} & \text{if } k = 1, \\ \frac{3}{8} & \text{if } k = 2, \\ \frac{1}{8} & \text{if } k = 3. \end{cases}$$

**Problem 7.** (2 points) Let $\mathcal{S}$ be a random variable representing the time it takes for a computer program to execute. The mean execution time is $100$ seconds, and the variance is $25$ seconds.

(a) Use Markov's inequality to estimate the probability that the program's execution time is at least 5 minutes.

**Solution.** We would like to estimate the probability that $\mathcal{S}$ is at least 5 minutes (300 seconds). Using Markov's inequality:

$$P(\mathcal{S} \geq 300) \leq \frac{\mathbb{E}(\mathcal{S})}{300} = \frac{100}{300} = \frac{1}{3}.$$

(b) Use Chebyshev's inequality to give the lower bound on the probability that the program's execution time is within 15 seconds of the mean.

**Solution.** Let $k = \frac{15}{\sigma} = \frac{15}{\sqrt{25}} = 3$. Using Chebyshev's inequality:

$$P(|\mathcal{S} - 100| \geq 15 = 3 \cdot 5) \leq \frac{1}{3^2} = \frac{1}{9}.$$

Therefore, the lower bound on the probability that $\mathcal{S}$ is within 15 seconds of the mean, which is $P(|\mathcal{S} - 100| < 15) = 1 - P(|\mathcal{S} - 100| \geq 15) \geq 1 - \frac{1}{9} = \frac{8}{9}$.