

# MATH *007* A

## Lecture 9

### Product, Quotient and Chain Rules

# This Week's Assignments

- **Homework 4.4, 4.5:** Due on *Monday* 10/27, 11:59 PM.

# Outline

- 1 Product Rule
- 2 Quotient Rule
- 3 Composition of Functions: a Reminder
- 4 Chain Rule

# Product Rule

The **product rule** states that if  $f(x) = u(x)v(x)$  is a product of two functions, then

$$f'(x) = u'(x)v(x) + u(x)v'(x).$$

## Example

Let  $f(x) = (x^2 + x) \cdot \left(5\sqrt{x} + \frac{3}{x^2}\right)$ . Then the product rule and power rule give

$$\begin{aligned} f'(x) &= (x^2 + x)' \cdot \left(5\sqrt{x} + \frac{3}{x^2}\right) + (x^2 + x) \cdot \left(5\sqrt{x} + \frac{3}{x^2}\right)' = \\ &= (2x + 1) \left(5x^{1/2} + 3x^{-2}\right) + (x^2 + x) \left(\frac{5}{2}x^{-1/2} - 6x^{-3}\right) \end{aligned}$$

## Exercise 9.1

### Exercise

Use the product rule to compute the derivative of

$$f(x) = (3x^5 - 2x^2 + 1) \cdot (3x^7 - x).$$

Then evaluate  $f'(0)$ .

# Solution

To differentiate, we use the product rule:

$$\begin{aligned} f'(x) &= (3x^5 - 2x^2 + 1)' \cdot (3x^7 - x) + (3x^5 - 2x^2 + 1) \cdot (3x^7 - x)' = \\ &= (15x^4 - 4x) \cdot (3x^7 - x) + (3x^5 - 2x^2 + 1) \cdot (21x^6 - 1) \end{aligned}$$

Next, evaluate at  $x = 0$ :

$$\begin{aligned} f'(0) &= (15 \cdot 0^4 - 4 \cdot 0)(3 \cdot 0^7 - 0) + (3 \cdot 0^5 - 2 \cdot 0^2 + 1)(21 \cdot 0^6 - 1) = \\ &= 1 \cdot (-1) = -1. \end{aligned}$$

# Quotient Rule

The **quotient rule** states that if  $f(x) = \frac{u(x)}{v(x)}$  where  $u$  and  $v$  are differentiable functions and  $v(x) \neq 0$ , then

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}.$$

## Example

Let  $f(x) = \frac{x^2+1}{3x^5-7x}$ . Then:

$$f'(x) = \frac{(2x)(3x^5 - 7x) - (x^2 + 1)(15x^4 - 7)}{(3x^5 - 7x)^2}$$

## Exercise 9.2

### Exercise

Use the quotient rule to compute the derivative of

$$f(x) = \frac{4x^2 + 1}{x^3 - 5x}.$$

Then evaluate  $f'(1)$ , and enter your answer as a decimal.

# Solution

Using the quotient and power rules, we compute:

$$f'(x) = \frac{(8x)(x^3 - 5x) - (4x^2 + 1)(3x^2 - 5)}{(x^3 - 5x)^2}.$$

Now evaluate at  $x = 1$ :

$$\begin{aligned} f'(1) &= \frac{8 \cdot 1 \cdot (1^3 - 5 \cdot 1) - (4 \cdot 1^2 + 1)(3 \cdot 1^2 - 5)}{(1^3 - 5 \cdot 1)^2} = \\ &= \frac{8 \cdot (-4) - 5 \cdot (-2)}{(-4)^2} = \frac{-32 + 10}{16} = \frac{-22}{16} = -1.375. \end{aligned}$$

## Composition of Functions: Motivational Example

Suppose you need to exchange dollars to euros, but you are in a bank that only deals with transactions involving pounds. The exchange rates are given by the following formulas: for every  $d$  dollars, you get  $f(d) = 0.7d$  pounds, and for every  $p$  pounds, you get  $g(p) = 1.2p - 3$  euros.

### Question

How many euros does the bank give for \$100?

**Answer.** To find out, we need to perform two exchanges:

$$\text{\$100} \xrightarrow{f} 0.7 \cdot 100 = \text{\pounds}70 \xrightarrow{g} 1.2 \cdot 70 - 3 = \text{\text{€}}81.$$

# Examples in Biology: DNA to Protein

Let  $x$  represent the initial information in DNA.

- $g(x)$ : Transcription—DNA is transcribed into mRNA.
- $h(g(x))$ : Translation—mRNA is translated into an amino acid chain (protein) in the ribosome.
- $f(h(g(x)))$ : Protein folding and modification—proteins fold and undergo post-translational modifications.

This sequence produces functional proteins essential for cellular processes.

# Neuron Signal Transmission

Neurons process information through a sequence of stages, culminating in a single “fire/not fire” decision:

- The neuron receives inputs from other neurons or stimuli.
- If the total input exceeds certain threshold, the neuron *fires*, generating an action potential.
- The action potential propagates along the axon toward the synapse.
- At the synapse, neurotransmitters are released, transmitting the signal to the next neuron.

Mathematically, this flow can be viewed as a composition of transformations:

$$k(f(h(g(x))))),$$

where each function represents a stage of signal processing in the neuron.

# From Neurons to AI

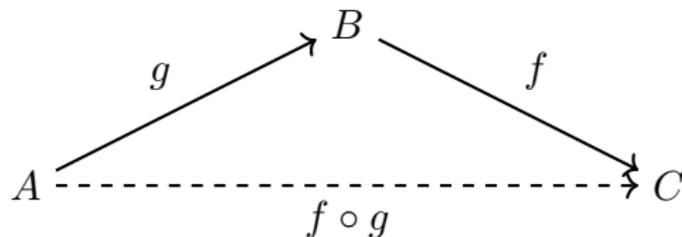
Just as biological neurons carry and transform signals through the brain, artificial neural networks—like the ones behind ChatGPT—process and transform information through many layers.

- Each artificial neuron computes a weighted sum of its inputs and applies a nonlinear *activation function*, which decides whether the neuron fires.
- Artificial neurons are arranged in layers. The output of one layer of neurons becomes the input to the next, forming a composition of functions.
- *Activation functions* play the role of biological thresholds, determining whether the signal is passed forward.
- Learning corresponds to adjusting the *weights*, modulating how strongly signals are transmitted.

# Formal Definition

## Definition

Let  $A$ ,  $B$ , and  $C$  be three sets, and  $g : A \rightarrow B$  and  $f : B \rightarrow C$  be two functions between them. The **composition** of  $f$  and  $g$  is a function  $f \circ g : A \rightarrow C$  obtained by successive applications of  $g$  and  $f$ , where  $f$  is applied to the result of the application of  $g$ .



## Remark

There are two common notations for composition:  $f(g(x))$  and  $(f \circ g)(x)$ . In both cases, the functions are applied from right to left, meaning you first apply  $g$  to  $x$  and then apply  $f$  to the result of  $g(x)$ .

# Example

## Example

- 1 Consider two functions  $f(x) = \sqrt{x - 5}$  and  $g(x) = 7 - x^2$ . Then  
 $(f \circ g)(x) = \sqrt{(7 - x^2) - 5} = \sqrt{2 - x^2}$  and  
 $(g \circ f)(x) = 7 - (\sqrt{x - 5})^2 = 7 - (x - 5) = 12 - x$ .
- 2 Consider two functions  $f(x) = |x|$  and  $g(x) = x - 3$ . Then

$$(f \circ g)(x) = |x - 3|, \quad \text{and} \quad (g \circ f)(x) = |x| - 3.$$

We can see that the order in which we apply functions affects the outcome. In other words, **composition of functions is not commutative**.

## Exercise 9.3

### Exercise

Evaluate the following compositions at the given points:

- 1 Let  $f(x) = x^2 + 1$  and  $g(x) = |x - 3|$ . Find  $(f \circ g)(5)$ .
- 2 Let  $f(x) = 2x - 4$  and  $g(x) = x^2$ . Find  $(g \circ f)(-1)$ .

# Solution

## Solution.

- ① For  $(f \circ g)(5)$ , we first find  $g(5)$  and then apply  $f$  to the result.

$$g(5) = |5 - 3| = 2.$$

Then,

$$f(g(5)) = f(2) = 2^2 + 1 = 4 + 1 = 5$$

Thus,  $(f \circ g)(5) = 5$ .

- ② For  $(g \circ f)(-1)$ , we first find  $f(-1)$  and then apply  $g$  to the result.

$$f(-1) = 2(-1) - 4 = -2 - 4 = -6.$$

Then,

$$g(f(-1)) = g(-6) = (-6)^2 = 36$$

Thus,  $(g \circ f)(-1) = 36$ .

# Chain Rule: Derivative of $(f \circ g)(x)$

Suppose:

- $y = g(x)$  - an "inner" function,
- $z = f(y)$  - an "outer" function.

Then  $z = f(g(x))$  - a composition of functions.

The rate at which  $z$  changes with respect to  $x$  is given by

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

## Examples: Apply the Chain Rule!

### Example

- ① Find the derivative of  $f(x) = (5x - 7)^4$ .

Let  $u = 5x - 7$ , then  $f(x) = u^4$ , so

$$f'(x) = 4(5x - 7)^3 \cdot (5x - 7)' = 4(5x - 7)^3 \cdot 5 = 20(5x - 7)^3.$$

- ② Find the derivative of  $g(x) = \sqrt{3x^2 + 5x - 27}$ .

Let  $u = 3x^2 + 5x - 27$ , then  $g(x) = u^{1/2}$ , so

$$\begin{aligned} g'(x) &= \frac{1}{2}(3x^2 + 5x - 27)^{-1/2} \cdot (3x^2 + 5x - 27)' = \\ &= \frac{1}{2}(3x^2 + 5x - 27)^{-1/2} \cdot (6x + 5). \end{aligned}$$

## Exercise 9.4

### Exercise

Compute the derivative of  $f(x) = \frac{4}{3x^2 - \sqrt{x}}$  and evaluate it at  $x = 1$ .

## Solution

We write the function  $f(x) = \frac{4}{3x^2 - \sqrt{x}}$  as a composition:

$$f(x) = \frac{4}{u(x)} \quad \text{where} \quad u(x) = 3x^2 - \sqrt{x}.$$

Then, using the chain rule:

$$f'(x) = 4 \cdot (-1) \cdot u(x)^{-2} \cdot u'(x) = \frac{-4}{(3x^2 - \sqrt{x})^2} \cdot (6x - 0.5x^{-1/2}).$$

Now plug in  $x = 1$ :

$$\begin{aligned} f'(1) &= \frac{-4}{(3 \cdot 1^2 - \sqrt{1})^2} \cdot (6 \cdot 1 - 0.5 \cdot 1^{-1/2}) = \\ &= \frac{-4}{(3 - 1)^2} \cdot \left(6 \cdot 1 - \frac{1}{2}\right) = \frac{-4}{4} \cdot 5.5 = -5.5. \end{aligned}$$

Chain Rule:   $\rightarrow$    $\rightarrow$  

- $y = \text{Jerry}(\text{Cheese})$ : position of Jerry as a function of cheese location,
- $z = \text{Tom}(\text{Jerry})$ : position of Tom as a function of Jerry's position.

Position of Tom as a function of cheese location is given by the composition of functions:   $\left( \text{Jerry}(\text{Cheese}) \right)$ .

Chain rule tells us how Tom's position changes as the cheese moves:

$$\text{Tom}(\text{Jerry}(\text{Cheese}))' = \text{Tom}'(\text{Jerry}(\text{Cheese})) \cdot \text{Jerry}'(\text{Cheese}).$$

### Remark

This is the rate at which Tom's position changes as the cheese moves - i.e., how sensitive Tom is to the cheese, via Jerry.