Lecture 8

Quadratic functions

MATH 0200

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Outline

Zeros of quadratic functions

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Definition

A **zero** of a function $f(x)$ is a number d with $f(d) = 0$ (the points where the graph of $f(x)$ intersects the x-axis).

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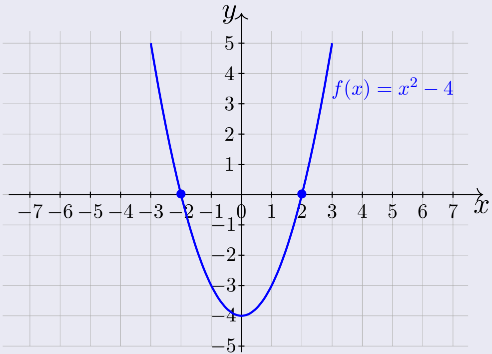

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Step 1. Rewrite bx, the linear term of $f(x)$ as

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Step 2. Add and subtract  ^{2}:

$f(x) = x^2 + 2 \cdot$  $\cdot\, x +$  $^2 -$  $^2 + c =$

$($ $x +$  $)^2 -$  $^2 + c.$

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We compute $(x-1)^2 - 1 - 8 = 0 \Leftrightarrow (x-1)^2 = 9 \Leftrightarrow x - 1 = \pm\sqrt{9} = \pm 3 \Leftrightarrow x = \pm 3 + 1 \Leftrightarrow x = 4$ or $x = -2$.

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- $f(-2) = (-2)^2 - 2 \cdot (-2) - 8 = 4 + 4 - 8 = 0 \quad \checkmark$
- $f(4) = 4^2 - 2 \cdot 4 - 8 = 16 - 8 - 8 = 0 \quad \checkmark$

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Example

We find the zeros of $f(x) = 3x^2 - 18x - 21$. The discriminant of $f(x)$ is $D = 18^2 - 4 \cdot 3 \cdot (-21) = 324 + 252 = 576 = 24^2$, and the roots are $(18 + 24)/6 = 7$ and $(18 - 24)/6 = -1$.

Parabola

The graph of a quadratic function $f(x) = ax^2 + bx + c$ is a parabola. It has a **vertex**, point $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$, and a **directrix** (line of symmetry), vertical line $x = -\frac{b}{2a}$.

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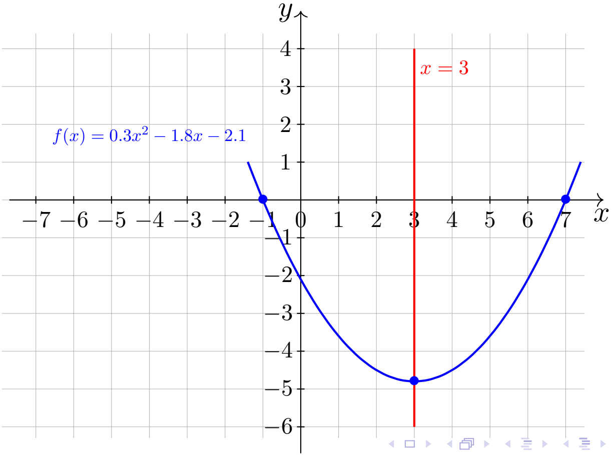

Question

Find the coordinates of the vertex of parabola given by equation $f(x) = -0.5x^2 + 7x - 4$.

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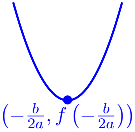

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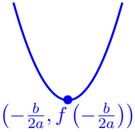

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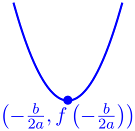

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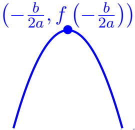

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Consider two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$. The **distance** between P and Q is the length of the line segment connecting these points. It is given by the formula

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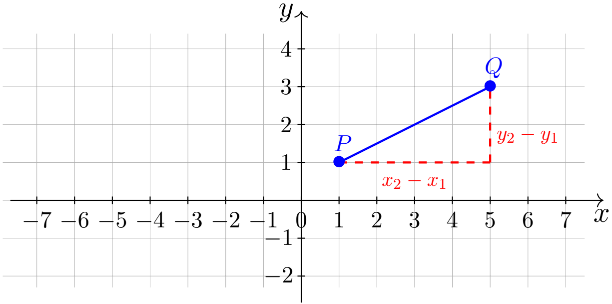

Question

What is the distance between the points $(-3, 1)$ and $(5, 7)$?

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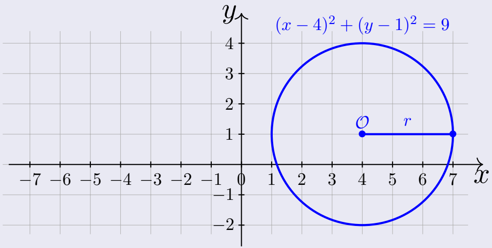

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We complete the squares:

$$x^2 - 6x + y^2 + 10y = 15 \Leftrightarrow$$
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The center is $(3, -5)$ and the radius is equal to 7.