MATH 007 A

Lecture 8

Practical Applications and Product Rule

This Week's Assignments

- **Homework 4.2, 4.3:** Due on *Monday* 10/20, 11:59 PM.
- **Microtutorials 2 and 3:** Due on *Monday* 10/20, 11:59 PM.
- **Quiz 2:** complete the quiz during your group's Discussion session.

Outline

Why Functions and Derivatives?

In many real-world problems, especially in biology and chemistry, quantities depend on one (or more) variable(s).

- The rate of an enzymatic reaction may depend on both temperature and substrate concentration.
- The pressure of a gas depends on both its volume and temperature, etc

To analyze how one variable influences the outcome while others are held fixed, we use derivatives with respect to different variables (also known as partial derivatives).

Example from Chemistry: Ideal Gas Law

Consider the Ideal Gas Law:

$$P = \frac{nRT}{V},$$

where:

- P is pressure,
- T is temperature,
- V is volume,
- n, R are constants.

We treat P as a function of T and V: $P(T, V) = \frac{nRT}{V}$.
The partial derivatives $\frac{\partial P}{\partial T} = \frac{nR}{V}$, and $\frac{\partial P}{\partial V} = -\frac{nRT}{V^2}$ measure

- how pressure changes with temperature (at fixed volume) and
- how pressure changes with volume (at fixed temperature).

Exercise 8.1

Exercise

Let $T = 300\,\mathrm{K}$, $n = 1\,\mathrm{mol}$, and $R = 0.08$. Find the rate of change of pressure with respect to volume when the volume is $V = 2\,\mathrm{L}$.

Solution

Let $T = 300\,\mathrm{K}$, $n = 1\,\mathrm{mol}$, and $R = 0.08$. Find the rate of change of pressure with respect to volume when the volume is $V = 2\,\mathrm{L}$.

Solution. We compute the derivative

$$\frac{\partial P}{\partial V} = \left(\frac{nRT}{V}\right)' = nRT\left(V^{-1}\right)' = -\frac{nRT}{V^2}$$

Substitute values:

$$\frac{\partial P}{\partial V} = -\frac{1 \cdot 0.08 \cdot 300}{2^2} = -\frac{24}{4} = -6\,\frac{\mathrm{atm}}{\mathrm{L}}.$$

Remark

The negative sign means pressure decreases as volume increases-consistent with Boyle's law.

Example from Biology: Enzyme Kinetics

In cell biology and biochemistry, many important chemical reactions are sped up by special proteins called enzymes. The speed (or rate) of these reactions depends on how much of the reacting molecule—called the *substrate*—is present.

A widely used model for this behavior is the **Michaelis–Menten equation**:

$$v = \frac{V_{\max} S}{K_M + S},$$

where:

- v: rate of reaction,
- S: substrate concentration,
- $V_{\max}$: maximum possible rate,
- K_M: Michaelis constant (affinity of enzyme for substrate).

The derivative $\frac{\partial v}{\partial S}$ helps to determine how sensitive a biological process is to changes in substrate availability.

Role of Differential Equations

So far, we have discussed partial derivatives. But often, we are interested in how systems evolve over time. This brings us to **differential equations**, which describe how variables change with time or other parameters.
Key idea: differential equations let us predict how biological or chemical systems evolve over time.

Newton's Law of Cooling

Suppose a hot object cools in a room at constant temperature.

The rate of change of the object's temperature is proportional to the difference between its temperature and the ambient temperature:

$$\frac{dT}{dt} = -k(T - T_{\text{env}})$$

where:

- $T(t)$ is the temperature of the object at time t,
- T_{env} is the surrounding (room) temperature,
- $k > 0$ is a constant that depends on the material/environment.

Solving this differential equation produces the formula

$$T(t) = T_{\text{env}} + (T_0 - T_{\text{env}})e^{-kt}.$$

Remark

This equation appears in both physics and biology when modeling temperature regulation.

Drug Elimination from the Bloodstream

Let $C(t)$ denote the concentration of a drug in the bloodstream at time t. A common model assumes the drug is eliminated at a rate proportional to its current concentration:

$$\frac{dC}{dt} = -kC,$$

where $k > 0$ is the elimination rate constant (depends on drug and patient).
Solving this differential equation produces the formula

$$C(t) = C_0 e^{-kt}.$$

This equation is used in pharmacokinetics to determine dosage timing and drug effectiveness.

Remark

The same equation appears in modeling radioactive decay, bacterial death, and many other natural processes.

Exercise 8.2

Exercise

Suppose a drug is administered such that the initial concentration in the bloodstream is $C_0 = 90\,\text{mg/L}$, and the elimination rate constant is $k = 0.2$ mg per hr.

1. Find the concentration $C(t)$ at time $t = \ell n(3^{10})$ hours.
2. What happens to the concentration as $t \to \infty$?

Solution

We substitute the given values of the constants $C_0 = 90$ and $k = 0.2$ into the equation to obtain $C(t) = 90e^{-0.2t}$.
At $t = \ell n(3^{10}) = 10\ell n(3)$ the concentration is

$$C(10\ell n(3)) = 90e^{-2\ell n(3)} = 90 \cdot 3^{-2} = 90 \cdot \frac{1}{9} = 10\text{mg/L}.$$

As $t \to \infty$:

$$\lim_{t\to\infty} C(t) = \lim_{t\to\infty} (90e^{-0.2t}) = 90 \lim_{t\to\infty} (e^{-0.2t}) = 0.$$

Remark

The drug concentration approaches zero over time, meaning that without additional doses, the drug is eventually eliminated from the bloodstream. However, at each concrete moment in time, there is still a positive (though possibly very small) amount of the drug present.

Product Rule

The **product rule** states that if $f(x) = u(x)v(x)$ is a product of two functions, then

$$f'(x) = u'(x)v(x) + u(x)v'(x).$$

Example

Let $f(x) = x^2 \cdot \cos(x)$. Then the product rule gives

$$\begin{aligned} f'(x) &= (x^2)' \cdot \cos(x) + x^2 \cdot (\cos(x))' = 2x \cdot \cos(x) + x^2 \cdot (-\sin(x)) \\ &= 2x \cdot \cos(x) - x^2 \cdot \sin(x). \end{aligned}$$

Exercise 8.3

Exercise

Use the product rule to compute the derivative of

$$f(x) = \frac{\sqrt{x}}{\sqrt{\pi}} \cdot \sin(x).$$

Then evaluate $f'(\pi)$.

Solution

To differentiate, we use the product rule:

$$f'(x) = \frac{1}{\sqrt{\pi}}\left((\sqrt{x})' \cdot \sin(x) + \sqrt{x} \cdot (\sin(x))'\right) =$$

$$\frac{1}{\sqrt{\pi}}\left(\frac{1}{2\sqrt{x}} \cdot \sin(x) + \sqrt{x}\cos(x)\right).$$

Next we evaluate $f'(x)$ at $x = \pi$:

$$f'(\pi) = \frac{1}{\sqrt{\pi}}\left(\frac{1}{2\sqrt{\pi}} \cdot \sin(\pi) + \sqrt{\pi}\cos(\pi)\right) = \frac{1}{\sqrt{\pi}}\left(0 - \sqrt{\pi}\right) =$$

$$-\frac{\sqrt{\pi}}{\sqrt{\pi}} = -1.$$