MATH 007 A

Lecture 7

Properties of Derivatives and Power Rule

This Week's Assignments

- **Homework 4.2, 4.3:** Due on *Monday* 10/20, 11:59 PM.
- **Microtutorials 2 and 3:** Due on *Monday* 10/20, 11:59 PM.
- **Quiz 2:** complete the quiz during your group's Discussion session.

Outline

1. Continuity vs. Differentiability

2. Basic Properties of Derivatives

3. Power Rule

Continuity vs. Differentiability

A function f is said to be **differentiable** at a point a if the following limit exists:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

In other words, the function must have a well-defined tangent line at that point.

Even if a function is continuous at a point, it might not be differentiable there.

Example

Consider $f(x) = |x|$. This function is continuous at $x = 0$, but it is not differentiable at that point because the left and right derivatives do not agree:

$$\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^-} (-x)' = \lim_{x \to 0^-} (-1) = -1;$$

$$\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} (x)' = \lim_{x \to 0^+} (1) = 1.$$

Exercise 7.0

Exercise

Determine for which values of x the function $h(x) = |4 - x^2|$ is differentiable.

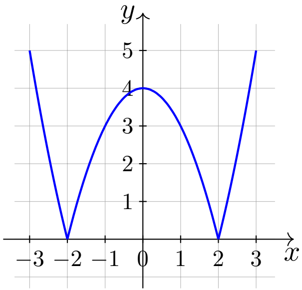
Solution. First, observe that $h(x) = |4 - x^2|$ can be written piecewise:

$$h(x) = \begin{cases} 4 - x^2 & \text{if } x^2 \le 4 \Leftrightarrow -2 \le x \le 2, \\ x^2 - 4 & \text{if } x^2 > 4 \Leftrightarrow x < -2 \text{ or } x > 2. \end{cases}$$

The transition happens at $x = -2$ and $x = 2$. The function is differentiable everywhere except $x = \pm 2$:

$$\lim_{x \to -2^-} h'(x) = \lim_{x \to -2} (x^2 - 4)' = \lim_{x \to -2} 2x = -4,$$
$$\lim_{x \to -2^+} h'(x) = \lim_{x \to -2} (4 - x^2)' = \lim_{x \to -2} (-2x) = 4.$$

Since the left and right derivatives are not equal, h is not differentiable at $x = -2$. Similarly, at $x = 2$, the one-sided derivatives also disagree (check it!), so h is not differentiable at $x = 2$ either.



Basic Properties of Derivatives

Let $f(x)$ and $g(x)$ be differentiable functions.

- $(cf(x))' = c \cdot f'(x)$ for any constant c;
- $(f(x) + g(x))' = f'(x) + g'(x)$;
- $(f(x) - g(x))' = f'(x) - g'(x)$.

These follow directly from the corresponding limit rules.

Power Rule

The **power rule** states that if $f(x) = x^n$, then $f'(x) = nx^{n-1}$ for any real number n.

Example

1. Let $f(x) = x^3$. Then $f'(x) = 3x^2$.
2. Let $\ell(x) = \sin(0.17\pi)\, x^4 - 0.5e^{\pi}\, x^2$. Then, applying the power rule:
   $\ell'(x) = \sin(0.17\pi) \cdot 4x^3 - 0.5e^{\pi} \cdot 2x = 4\sin(0.17\pi)x^3 - e^{\pi}x.$

Remark

No matter how scary a constant (number) looks, it is still just a number - treat it like any other coefficient.

Exercise 7.1

Exercise

Use the power rule to compute the derivative of the function $f(x) = \frac{1}{\sqrt[4]{x^7}} + x^5$. Then evaluate $f'(1)$ and enter your answer as a decimal.

Solution

Use the power rule to compute the derivative of the function $f(x) = \frac{1}{\sqrt[4]{x^7}} + x^5$. Then evaluate $f'(1)$.
We first rewrite the function using exponents:

$$f(x) = x^{-7/4} + x^5.$$

Now apply the power rule:

$$f'(x) = -\frac{7}{4} \cdot x^{-11/4} + 5x^4.$$

Evaluate at $x = 1$:

$$f'(1) = -\frac{7}{4} \cdot 1^{-11/4} + 5 \cdot 1^4 = -\frac{7}{4} + 5 = 3\frac{1}{4} = 3.25.$$

Differentiation with Constants

Differentiate the function $g(\zeta) = r\zeta\left(\frac{\zeta}{C} - 10\right)$ with respect to ζ. Assume r and C are positive constants.

First, expand the expression:

$$g(\zeta) = r\zeta \cdot \left(\frac{\zeta}{C} - 10\right) = r\left(\frac{\zeta^2}{C} - 10\zeta\right).$$

Now compute the derivative:

$$g'(\zeta) = r\left(\left(\frac{\zeta^2}{C}\right)' - (10\zeta)'\right) = r\left(\frac{2\zeta}{C} - 10\right).$$

Remark

Constants like r and C are treated as fixed values when differentiating with respect to ζ.

Exercise 7.2

Exercise

Consider the function $\frac{5U}{V}$.

1. Compute the derivative with respect to U, treating V as a constant. Then evaluate your result at $U = 2$, $V = 1$.
2. Compute the derivative with respect to V, treating U as a constant. Then evaluate your result at $U = 2$, $V = 1$.

Solution

Consider the function $\frac{5U}{V}$.

1. Compute the derivative with respect to U, treating V as a constant. Then evaluate your result at $U = 2$, $V = 1$.

   The derivative with respect to U is $\frac{5 \cdot 1}{V}$. Evaluating at $U = 2$, $V = 1$:
   $$\frac{5 \cdot 1}{1} = 5.$$

2. Compute the derivative with respect to V, treating U as a constant. Then evaluate your result at $U = 2$, $V = 1$.

   The derivative with respect to V is $-\frac{5U}{V^2}$. Evaluating at $U = 2$, $V = 1$:
   $$-\frac{5 \cdot 2}{1^2} = -10.$$

Power Rule: Where Does It Come From?

Let $f(x) = x^n$. To find the derivative, we need to compute the limit:

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}.$$

Let's expand just the first few terms of $(x+h)^n$:

$$(x+h)^n = x^n + nx^{n-1}h + (\text{smaller terms involving } h^2, h^3, \dots).$$

So, $\frac{(x+h)^n - x^n}{h} = \frac{nx^{n-1}h + (\text{smaller terms})}{h} = nx^{n-1} + \text{terms involving } h.$
Taking the limit as $h \to 0$:

$$f'(x) = \lim_{h \to 0} \left(\frac{nx^{n-1}h + (\text{smaller terms})}{h} \right) = \lim_{h \to 0} (nx^{n-1} + \text{terms involving}$$

Power Rule: Where Does It Come From?

Question

Why is the coefficient of hx^{n-1} equal to n?

There are n ways to choose one of the n $x + h$ factors to contribute an h, while the rest contribute x's.