MATH 007 A

Lecture 6

Tangent Lines and Properties of Derivatives

This Week's Assignments

- **Homework 3.4, 4.1** Due on *Monday* 10/13, 11:59 PM.
- **Microtutorial 1: Salt Concentration** Due on *Monday* 10/13, 11:59 PM.
- **Quiz 1:** complete the quiz during your group's Discussion session.

Outline

Tangent Line and Derivative

The **tangent line** to the graph of a function $f(x)$ at a point $x = a$ is the best linear approximation of the function near that point.
It passes through the point $(a, f(a))$ and has slope $f'(a)$, the derivative at that point.

Equation of the tangent line:

$$y = f(a) + f'(a)(x - a)$$

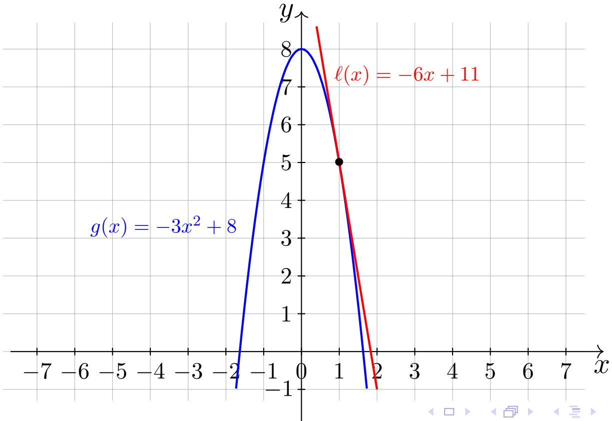
This is just the point-slope form of a line with slope $f'(a)$ through $(a, f(a))$.

Tangent Line to $g(x) = -3x^2 + 8$ at $x = 1$

Recall from our earlier example that for $g(x) = -3x^2 + 8$ we have

$$g(1) = 5 \text{ and } g'(1) = -6.$$

So the tangent line at the point $(1, 5)$ has slope -6, and its equation is given by $y - 5 = -6(x - 1) \Leftrightarrow y = -6x + 6 + 5 \Leftrightarrow y = -6x + 11$.



General Formula for the Equation of Tangent Line

Given a function $f(x)$ (which has a derivative at $x = a$), the tangent line to the graph at $x = a$ is:

$$y = f(a) + f'(a)(x - a)$$

This is useful for

- approximating the value of $f(x)$ near $x = a$;
- understanding the local behavior of the function;
- constructing linear models in applied problems.

Exercise 6.1

Exercise

Let $f(x) = 5x^2 + 2x$.

1. Compute the derivative $f'(0)$.
2. Determine the slope of the tangent line to the graph of f at $x = 0$.
3. Find the equation of the tangent line to $f(x)$ at $x = 0$, and compute its value at $x = 1$.

Solution

We begin by finding the derivative:

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h} =$$
$$\frac{5h^2 + 2h - (5 \cdot 0^2 + 2 \cdot 0)}{h} = \lim_{h \to 0} (5h + 2) = 2.$$

The slope of the tangent line to the graph of f at $x = 0$ is equal to 2, the value of the derivative at that point.

The y-coordinate of the point of tangency is $f(0) = 5 \cdot 0^2 + 2 \cdot 0 = 0$. We get the equation

$$y - 0 = 2(x - 0) \Leftrightarrow y = 2x.$$

Finally, to find the value of the tangent line at $x = 1$, we compute $y = 2 \cdot 1 = 2$.

Example

Find the equation of the tangent line to the graph of $f(x) = \frac{2}{x+4}$ at $(1, 0.4)$.

We begin by finding the derivative using the definition:

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{\frac{2}{(1+h)+4} - \frac{2}{5}}{h} = \lim_{h\to 0} \frac{\frac{2}{5+h} - \frac{2}{5}}{h} = \lim_{h\to 0} \frac{2\left(\frac{1}{5+h} - \frac{1}{5}\right)}{h}.$$

Use common denominator to simplify the numerator:

$$\lim_{h\to 0} \frac{2\left(\frac{5-(5+h)}{(5+h)\cdot 5}\right)}{h} = \lim_{h\to 0} \frac{2\left(\frac{-h}{5(5+h)}\right)}{h} = \lim_{h\to 0} \frac{-2}{5(5+h)} = -\frac{2}{25}.$$

The slope of the tangent line at $x = 1$ is $-\frac{2}{25}$, and the point of tangency is $(1, 0.4)$. Hence, the equation of the tangent line is

$$y - 0.4 = -\frac{2}{25}(x-1) \quad \Leftrightarrow \quad y = -\frac{2}{25}(x-1) + 0.4.$$

What Does the Derivative Tell Us?

Recall that the derivative at a point equals the slope of the tangent line at that point.

- If $f'(x) > 0$, then f is **increasing** near x.
- If $f'(x) < 0$, then f is **decreasing** near x.
- If $f'(x) = 0$, then the tangent line is **horizontal**.

When $f'(x) = 0$, the point x is called a **critical point**. It could be:

- a **local maximum**,
- a **local minimum**, or
- a **point of inflection**.

To determine which one it is, we will use the **second derivative test**-more on that soon!

Key Takeaways – Disneyland Edition