MATH 007 A

Lecture 5

Derivatives: Let's Get Acquainted

This Week's Assignments

- **Homework 3.4, 4.1** Due on *Monday* 10/13, 11:59 PM.
- **Microtutorial 1: Salt Concentration** Due on *Monday* 10/13, 11:59 PM.
- **Quiz 1:** complete the quiz during your group's Discussion session.

Outline

Why Derivatives?

Derivatives help us understand how a function behaves at a given point:

- Where is the function increasing or decreasing?
- Where are the local maximum and minimum values?
- What is the function's rate of change at a specific point?

In applications, derivatives often have concrete interpretations:

- **Marginal cost** in economics: the derivative of the cost function $C(x)$ gives the approximate increase in cost from producing one more unit when x units are already produced.
- **Population growth rate** in biology: the derivative of the population size with respect to time gives the instantaneous rate of growth or decline.

From Average to Instantaneous Rate of Change

Recall the Average Rate of Change (AROC) in a function $f(x)$ over between points $x = a$ and $x = a + h$ is given by the formula

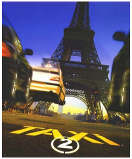
$$\text{AROC} = \frac{f(a+h) - f(a)}{h}.$$

To find how fast f is changing *at exactly* $x = a$, we take the limit as $h \to 0$. This gives the **instantaneous rate of change**, also known as the **derivative**:

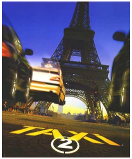
$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Analogy: Derivative as Instantaneous Speed

Think of a car driving along a road. The **average speed** between two points is (distance traveled)/(time taken) the Average Rate of Change. But the **instantaneous speed** at a specific moment is what a **speed radar gun** shows - and that's exactly what the **derivative** captures:



$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



In Luc Besson's high-octane movie *Taxi 2*, there's a famous radar scene where the car is going... a little faster than legal limits, see the clip here.

The Taxi films (especially the first two) are over-the-top action-comedies made in France, starring a pizza-delivery-guy-turned-speedster and a chronically terrified cop (good for math breaks)!

Definition of the Derivative

The derivative of a function $f(x)$ at a point $x = a$ is defined as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

This gives the instantaneous rate of change of the function at the specific point $x = a$.
If we allow the point a to vary, this definition leads to a new function, called the *derivative* and denoted by $f'(x)$, defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

This function tells us the rate of change of $f(x)$ at any point x in its domain (where the limit exists). An alternative notation for the derivative is $\frac{d}{dx} f(x)$, which reads as *the derivative of $f(x)$ with respect to x*. Thus, $f'(x) = \frac{d}{dx} f(x)$.

Derivative of a Linear Function

Example

Let $f(x) = 3 - 2x$. Find $f'(1)$ and $f'(3)$.
Using the limit definition of the derivative, we have:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{[3 - 2(a+h)] - [3 - 2a]}{h} =$$
$$\lim_{h \to 0} \frac{-2a - 2h + 2a}{h} = \lim_{h \to 0} \frac{-2h}{h} = -2.$$

This result tells us that the derivative is the same for all values of a, meaning the rate of change of the function $f(x) = 3 - 2x$ is constant. Since $f(x)$ is a linear function with constant slope -2, the tangent line at any point is the line itself, and the slope is always -2. In particular, $f'(1) = f'(3) = -2$.

Derivative of a Quadratic Function

Example

Let $g(x) = -3x^2 + 8$. Find $g'(1)$ using the definition.
We first compute

$$g(1+h) = -3(1+h)^2 + 8 = -3(1+2h+h^2) + 8 =$$
$$-3 - 6h - 3h^2 + 8 = 5 - 6h - 3h^2,$$
$$g(1) = -3 \cdot 1^2 + 8 = 5.$$

Therefore, $g'(1) = \lim_{h\to 0} \frac{g(1+h)-g(1)}{h} = \lim_{h\to 0} \frac{(5-6h-3h^2)-5}{h} =$
$\lim_{h\to 0} \frac{-6h-3h^2}{h} = \lim_{h\to 0}(-6-3h) = -6.$

Derivative of a Square Root Function

Example

Let $f(x) = \sqrt{6x}$. Find $f'(3)$ using the definition.

$$f'(3) = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0} \frac{\sqrt{6(3+h)}-\sqrt{6\cdot 3}}{h} =$$
$$\lim_{h\to 0} \frac{\sqrt{18+6h}-\sqrt{18}}{h}.$$

Multiply numerator and denominator by the conjugate:

$$\lim_{h\to 0} \frac{\sqrt{18+6h}-\sqrt{18}}{h} \cdot \frac{\sqrt{18+6h}+\sqrt{18}}{\sqrt{18+6h}+\sqrt{18}} = \lim_{h\to 0} \frac{(18+6h)-18}{h(\sqrt{18+6h}+\sqrt{18})} =$$
$$\lim_{h\to 0} \frac{6h}{h(\sqrt{18+6h}+\sqrt{18})} = \lim_{h\to 0} \frac{6}{\sqrt{18+6h}+\sqrt{18}} = \frac{6}{2\sqrt{18}} = \frac{3}{\sqrt{18}} =$$
$$\frac{3}{\sqrt{3^2\cdot 2}} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}.$$