

MATH *007^F* A

Lecture 4



with Limits

This Week's Assignments

- **Homework 3.4, 4.1** Due on *Monday* 10/13, 11:59 PM.
- **Microtutorial 1: Salt Concentration** Due on *Monday* 10/13, 11:59 PM.
- **Quiz 1:** complete the quiz during your group's Discussion session.

Outline

- 1 Rationalizing with the Conjugate
- 2 The Squeeze Theorem

Rationalizing with the Conjugate: an Example

Example

Compute $\lim_{x \rightarrow 0} \left(\frac{\sqrt{x^2+49}-7}{x^2} \right)$.

First we multiply (and divide) by $\sqrt{x^2+49} - 7$, the *conjugate* of $\sqrt{x^2+49} + 7$:

$$\begin{aligned} \frac{\sqrt{x^2+49}-7}{x^2} &= \frac{\sqrt{x^2+49}-7}{x^2} \cdot \frac{\sqrt{x^2+49}+7}{\sqrt{x^2+49}+7} = \\ \frac{(\sqrt{x^2+49}-7)(\sqrt{x^2+49}+7)}{x^2(\sqrt{x^2+49}+7)} &= \frac{(x^2+49)-49}{x^2(\sqrt{x^2+49}+7)} = \\ \frac{x^2}{x^2(\sqrt{x^2+49}+7)} &= \frac{1}{\sqrt{x^2+49}+7}. \end{aligned}$$

Next we compute $\lim_{x \rightarrow 0} \left(\frac{1}{\sqrt{x^2+49}+7} \right) = \frac{1}{\sqrt{0^2+49}+7} = \frac{1}{14}$.

Exercise 4.0

Exercise

Compute $\lim_{x \rightarrow 1} \left(\frac{\sqrt{x^2+3}-2}{x-1} \right)$.

Solution

Compute $\lim_{x \rightarrow 1} \left(\frac{\sqrt{x^2+3}-2}{x-1} \right)$.

First we multiply (and divide) by $\sqrt{x^2+3}+2$, the *conjugate* of $\sqrt{x^2+3}-2$:

$$\begin{aligned} \frac{\sqrt{x^2+3}-2}{x-1} &= \frac{\sqrt{x^2+3}-2}{x-1} \cdot \frac{\sqrt{x^2+3}+2}{\sqrt{x^2+3}+2} = \\ \frac{(\sqrt{x^2+3}-2)(\sqrt{x^2+3}+2)}{(x-1)(\sqrt{x^2+3}+2)} &= \frac{(x^2+3)-4}{(x-1)(\sqrt{x^2+3}+2)} = \\ \frac{x^2-1}{(x-1)(\sqrt{x^2+3}+2)} &= \frac{(x-1)(x+1)}{(x-1)(\sqrt{x^2+3}+2)} = \frac{x+1}{\sqrt{x^2+3}+2}. \end{aligned}$$

Next we compute $\lim_{x \rightarrow 1} \left(\frac{x+1}{\sqrt{x^2+3}+2} \right) = \frac{1+1}{\sqrt{1^2+3}+2} = \frac{2}{4} = \frac{1}{2}$.

The Squeeze Theorem

The *Squeeze Theorem*, helps us evaluate limits of functions that are difficult to compute directly.

Suppose we have three functions $g(x)$, $f(x)$, $h(x)$ defined near $x = a$, such that:

$$g(x) \leq f(x) \leq h(x) \quad \text{for all } x \text{ near } a$$

and if

$$\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L,$$

then the theorem tells us that $\lim_{x \rightarrow a} f(x) = L$.

The Squeeze Theorem

This theorem is useful when a function is "trapped" between two others that both approach the same limit:



\lim



Also known as the Sandwich Theorem!

From now on, we will use the symbol 🥪 for it.

Applying the Theorem

Example

Let's compute $\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right)$.

Question

Why can't we use the product rule?

The reason is that the limit $\lim_{x \rightarrow 0} \left(\sin\left(\frac{1}{x}\right)\right)$ does *not* exist (the sine function oscillates infinitely near zero).

Applying the Theorem

Instead, we can apply the  Theorem. We begin with the known bounds for the sine function:

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1.$$

Multiplying all parts of this inequality by x^2 (which is non-negative for all real x) gives:

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2.$$

Now take the limits:

$$\lim_{x \rightarrow 0} (-x^2) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (x^2) = 0.$$

So the  Theorem gives $\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

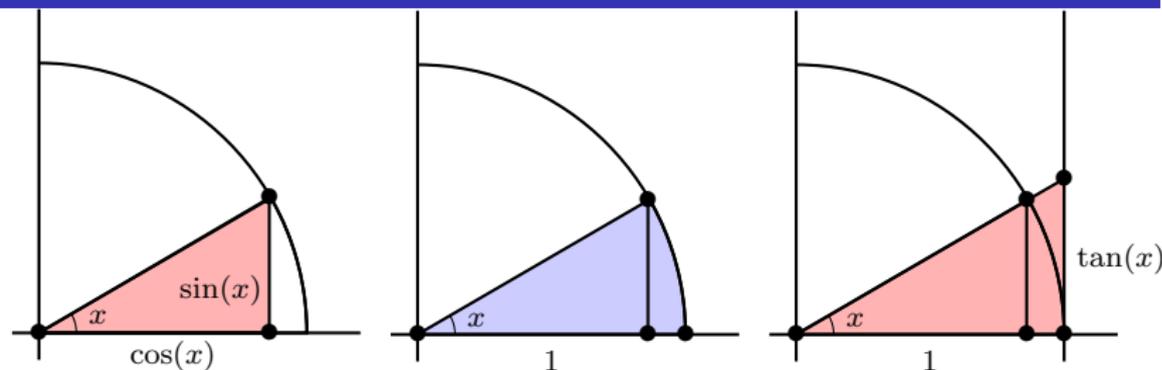
The Limit $\lim_{x \rightarrow 0} \left(\frac{\sin(x)}{x} \right)$

One of the most important trigonometric limits is

$$\lim_{x \rightarrow 0} \left(\frac{\sin(x)}{x} \right) = 1.$$

On the next slide, we outline how the  Theorem allows us to prove this result geometrically using area comparisons.

Idea of Proof



Since the inner triangle is contained in the circular sector, which is in turn contained in the outer triangle, their areas satisfy

$$0.5 \sin(x) \cos(x) \leq 0.5x \leq 0.5 \tan(x).$$

Now divide all parts of this inequality by $0.5 \sin(x)$ (valid since $\sin(x) > 0$ for small $x > 0$):

$$\cos(x) \leq \frac{x}{\sin(x)} \leq \frac{1}{\cos(x)}.$$

Idea of Proof

As $x \rightarrow 0^+$ and $\cos(x)$ is continuous at $x = 0$, we obtain

$$\lim_{x \rightarrow 0^+} (\cos(x)) = 1 \quad \text{and} \quad \lim_{x \rightarrow 0^+} \left(\frac{1}{\cos(x)} \right) = \frac{1}{1} = 1.$$

Thus, by the  Theorem:

$$\lim_{x \rightarrow 0^+} \left(\frac{x}{\sin(x)} \right) = 1.$$

Since $\frac{x}{\sin(x)}$ is an even function (i.e., $\frac{-x}{\sin(-x)} = \frac{x}{\sin(x)}$), the same limit holds as $x \rightarrow 0^-$, and we conclude:

$$\lim_{x \rightarrow 0} \left(\frac{x}{\sin(x)} \right) = 1 \quad \Rightarrow \quad \lim_{x \rightarrow 0} \left(\frac{\sin(x)}{x} \right) = 1.$$

Example

Example

Evaluate $\lim_{x \rightarrow 0} \left(\frac{\sin(5x)}{20x} \right)$.

Step 1. The standard limit we know is $\lim_{x \rightarrow 0} \left(\frac{\sin(x)}{x} \right) = 1$. To use it, we want to create an expression of the form $\frac{\sin(u)}{u}$.

Step 2. Notice that inside the sine we have $5x$, not just x , so we aim to match denominator to the argument of sine:

$$\frac{\sin(5x)}{20x} = \frac{1}{4} \cdot \frac{\sin(5x)}{5x}.$$

Step 3. Now we use the limit $\lim_{x \rightarrow 0} \left(\frac{\sin(5x)}{5x} \right) = 1$ to find

$$\lim_{x \rightarrow 0} \left(\frac{\sin(5x)}{20x} \right) = \frac{1}{4} \cdot \lim_{x \rightarrow 0} \left(\frac{\sin(5x)}{5x} \right) = \frac{1}{4}.$$

Exercise 4.1

Exercise

Evaluate $\lim_{x \rightarrow 0} \left(\frac{3x}{\sin(5x)} \right)$ and enter your answer as a decimal.

Solution

Evaluate $\lim_{x \rightarrow 0} \left(\frac{3x}{\sin(5x)} \right)$ and enter your answer as a decimal.

Rewrite the expression to isolate a known limit:

$$\frac{3x}{\sin(5x)} = \frac{3 \cdot \frac{5}{5}x}{\sin(5x)} = \frac{3}{5} \cdot \frac{5x}{\sin(5x)}.$$

Using that $\lim_{x \rightarrow 0} \left(\frac{5x}{\sin(5x)} \right) = 1$, we find

$$\lim_{x \rightarrow 0} \left(\frac{3x}{\sin(5x)} \right) = \frac{3}{5} \cdot \lim_{x \rightarrow 0} \left(\frac{5x}{\sin(5x)} \right) = \frac{3}{5} = 0.6.$$

Example

Evaluate $\lim_{x \rightarrow 0} \left(\frac{1 - \cos^2(6x)}{6x} \right)$.

Step 1. Use the trigonometric identity $\cos^2(6x) + \sin^2(6x) = 1$ to write

$$1 - \cos^2(6x) = \sin^2(6x).$$

Step 2. Use one of the basic properties of limits to obtain

$$\begin{aligned} \lim_{x \rightarrow 0} \left(\frac{1 - \cos^2(6x)}{6x} \right) &= \lim_{x \rightarrow 0} \left(\frac{\sin^2(6x)}{6x} \right) = \\ \lim_{x \rightarrow 0} \left(\sin(6x) \cdot \frac{\sin(6x)}{6x} \right) &= \lim_{x \rightarrow 0} (\sin(6x)) \lim_{x \rightarrow 0} \left(\frac{\sin(6x)}{6x} \right). \end{aligned}$$

Step 3. Use that limit $\lim_{x \rightarrow 0} \left(\frac{\sin(6x)}{6x} \right) = 1$ and $\lim_{x \rightarrow 0} \sin(6x) = 0$ to get the final answer:

$$\lim_{x \rightarrow 0} \left(\frac{1 - \cos^2(6x)}{6x} \right) = 0 \cdot 1 = 0.$$

The Limit $\lim_{x \rightarrow 0} \left(\frac{1 - \cos(x)}{x} \right) = 0$

Another trigonometric limit we will need to learn is

$$\lim_{x \rightarrow 0} \left(\frac{1 - \cos(x)}{x} \right) = 0.$$

Remark

This follows from the identity $1 - \cos(x) = 2 \sin^2 \left(\frac{x}{2} \right)$, and application of $\lim_{x \rightarrow 0} \left(\frac{\sin(x)}{x} \right) = 1$.

Example

Example

Evaluate $\lim_{x \rightarrow 0} \left(\frac{\cos(x)-1}{4x} \right)$.

We rewrite the limit:

$$\lim_{x \rightarrow 0} \left(\frac{\cos(x)-1}{4x} \right) = -\frac{1}{4} \cdot \lim_{x \rightarrow 0} \left(\frac{1-\cos(x)}{x} \right).$$

Using the known limit: $\lim_{x \rightarrow 0} \left(\frac{1-\cos(x)}{x} \right) = 0$,

we conclude that

$$\lim_{x \rightarrow 0} \left(\frac{\cos(x)-1}{4x} \right) = -\frac{1}{4} \cdot 0 = 0.$$