

MATH *007* A

Lecture 3

Limits and Continuous Functions

# This Week's Assignments

- **Homework 3.1-3.3:** Due on *Monday* 10/06, 11:59 PM.

# Outline

- 1 Continuous Functions
- 2 Behavior of a Rational Function as  $x \rightarrow \pm\infty$
- 3 Limits Involving Exponential Functions

# Continuous Functions: Informal Definition

In everyday terms, a continuous function is one whose graph can be drawn without lifting the pen off the paper—essentially, *continuously* moving the pen. It's like drawing a curve without any sudden jumps or interruptions.

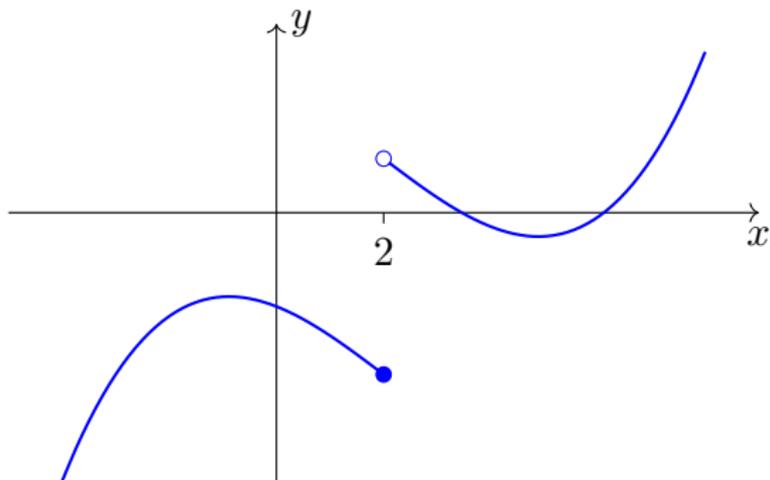
To illustrate, imagine you are drawing a function graph. If you get tired and lift the pen off the paper, then restart at a different point, the function would be considered discontinuous at that point. So, continuous functions are the ones you can draw without any breaks.

More conceptually, a function  $f(x)$  is **continuous at a point**  $x = a$  if the value of  $f(x)$  as  $x$  approaches  $a$  from the left coincides with the value as  $x$  approaches  $a$  from the right, and both are equal to the function's value at  $x = a$ . This is expressed as

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a)$$

If this condition is not satisfied, the function is said to be **discontinuous** at  $x = a$ .

Below is a graph illustrating a function discontinuous at  $x = 2$ :



## Example

Given a function defined by the formula

$$f(x) = \begin{cases} c - 0.5x, & x \leq -1, \\ 3 - x^2, & x > -1, \end{cases}$$

find the value of  $c$  that ensures the continuity of  $f(x)$  at the point  $x = -1$ .

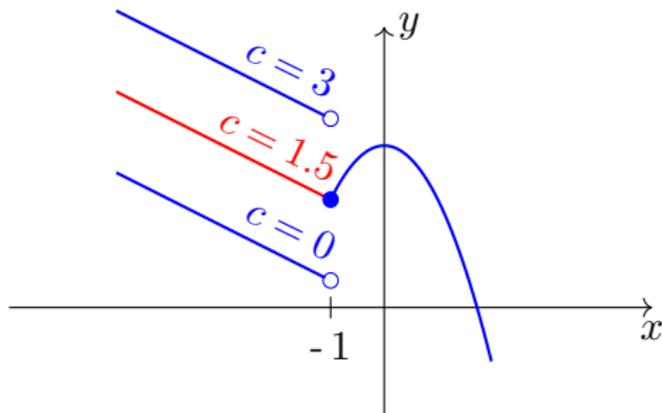
For  $f(x)$  to be continuous at  $x = -1$ , the values of  $f(x)$  as  $x$  approaches  $-1$  from the left and right must be equal. We compute:

$$\lim_{x \rightarrow -1^-} f(x) = c - 0.5 \cdot (-1) = c + 0.5 \quad (\text{using the formula for } x \leq -1),$$

$$\lim_{x \rightarrow -1^+} f(x) = 3 - (-1)^2 = 3 - 1 = 2 \quad (\text{using the formula for } x > -1).$$

Setting these expressions equal gives  $c + 0.5 = 2$ , which simplifies to  $c = 1.5$ .

# Graphs of $f(x)$ for Different Values of $c$



## Exercise 3.1

### Exercise

For the piecewise function

$$g(x) = \begin{cases} cx^2, & x < 2, \\ 8x + 8, & x \geq 2, \end{cases}$$

find  $c$  such that  $g(x)$  is continuous at  $x = 2$ .

# Solution

For the piecewise function

$$g(x) = \begin{cases} cx^2, & x < 2, \\ 8x + 8, & x \geq 2, \end{cases}$$

find  $c$  such that  $g(x)$  is continuous at  $x = 2$ .

We compute:

$$\begin{aligned} \lim_{x \rightarrow 2^-} g(x) &= c \cdot 2^2 = 4c, \\ \lim_{x \rightarrow 2^+} g(x) &= 8 \cdot 2 + 8 = 24. \end{aligned}$$

Setting  $4c = 24$  gives  $c = 6$ .

## Continuity: an Important Remark

Recall that using the limit notation, the definition of a function  $f(x)$  being continuous at a point  $x = a$  can be expressed as the equality of limits:  $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a)$ .

### Remark

In order to be continuous at a point  $x = a$ , the function  $f(x)$  must be defined at that point.

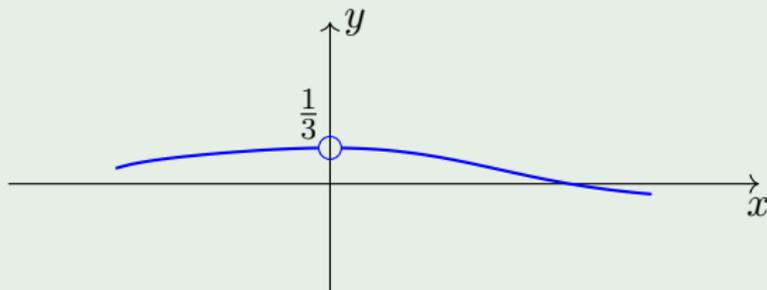
## Example

Consider  $g(x) = \frac{5x-x^3}{x^4+15x}$ . We compute:

$$\lim_{x \rightarrow 0^+} g(x) = \lim_{x \rightarrow 0^+} \frac{5x - x^3}{x^4 + 15x} = \lim_{x \rightarrow 0^+} \frac{5 - x^2}{x^3 + 15} = \frac{5 - 0^2}{0^3 + 15} = \frac{1}{3};$$

$$\lim_{x \rightarrow 0^-} g(x) = \lim_{x \rightarrow 0^-} \frac{5x - x^3}{x^4 + 15x} = \lim_{x \rightarrow 0^-} \frac{5 - x^2}{x^3 + 15} = \frac{5 - 0^2}{0^3 + 15} = \frac{1}{3}.$$

which coincide. However, it is important to note that  $f(x)$  is not defined at  $x = 0$  since this point falls outside the domain of  $f(x)$ . Therefore, the function is not continuous at  $x = 0$ . Such a scenario is often referred to as a *hole*.



# Continuity of Polynomial and Rational Functions

- Polynomial functions are continuous everywhere. That is, for any polynomial  $p(x)$  and any number  $a$ , we have:

$$\lim_{x \rightarrow a} p(x) = p(a).$$

- Rational functions are continuous everywhere they are defined. However, they are not defined at points where the denominator is zero. That is, if  $r(x) = \frac{p(x)}{q(x)}$ , then  $r(x)$  is continuous at all points where  $q(x) \neq 0$ .

# Computing Limits Using Continuity

If a function  $f(x)$  is continuous at  $x = a$ , then by definition:

$$\lim_{x \rightarrow a} f(x) = f(a).$$

This makes evaluating limits straightforward—simply substitute  $x = a$  into the function.

## Example

Evaluate the following limit:

$$\lim_{x \rightarrow 2} (3x^2 - 4x + 1).$$

Since the function is a polynomial, it is continuous everywhere, so we substitute  $x = 2$ :

$$3 \cdot 2^2 - 4 \cdot 2 + 1 = 3 \cdot 4 - 8 + 1 = 12 - 8 + 1 = 5.$$

# Limits of Rational Functions

Consider the rational function:

$$f(x) = \frac{2x^3 - x + 5}{x^2 - 3}.$$

To compute  $\lim_{x \rightarrow 2} f(x)$ , we check if the denominator is nonzero at  $x = 2$ :

$$2^2 - 3 = 4 - 3 = 1 \neq 0.$$

Since  $f(x)$  is continuous at  $x = 2$ , we substitute:

$$\frac{2 \cdot 2^3 - 2 + 5}{2^2 - 3} = \frac{2 \cdot 8 - 2 + 5}{4 - 3} = \frac{16 - 2 + 5}{1} = 19.$$

## Exercise 3.2

### Exercise

A more rigorous definition of continuity involves the epsilon-delta formalism, which is studied in more advanced calculus courses:

$$\forall \varepsilon > 0 \exists \delta > 0 : |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon.$$

On the Eiffel Tower, 72 names of French scientists, engineers, and mathematicians are engraved in recognition of their contributions. One of these individuals, whose name appears on the part of the tower pictured below, created the definition above. Identify this individual.



# Solution

The individual associated with the rigorous definition of continuity and limits is Augustin-Louis Cauchy. His pioneering work in mathematical analysis laid the foundation for the epsilon-delta formalism. Cauchy's name was inscribed on the Eiffel Tower in recognition of his significant contributions to mathematics.

## Behavior of a Rational Function as $x \rightarrow \pm\infty$

We will start with a few examples.

Consider the function  $f(x) = \frac{4x^7 - 2x^2 + 1}{x^7 + 2x + 2}$ .

Recall: the **leading term** of a polynomial is the term with the highest power of  $x$ .

- The leading term of the numerator is  $4x^7$ .
- The leading term of the denominator is  $x^7$ .

The behavior of a rational function as  $x \rightarrow \pm\infty$  depends only on the ratio of leading terms of the numerator and denominator:

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{4x^7}{x^7} = 4.$$

By the same reasoning, we obtain

$$\lim_{x \rightarrow -\infty} f(x) = 4.$$

# Examples

Consider the function  $g(x) = \frac{-2x^2 - 3x^5 + 5}{2x^3 + x + 1}$ .

- The leading term of the numerator is  $-3x^5$ .
- The leading term of the denominator is  $2x^3$ .

Notice that the degree of the numerator (5) is greater than the degree of the denominator (3). We compute

$$\lim_{x \rightarrow \infty} g(x) = \lim_{x \rightarrow \infty} \left( \frac{-3x^5}{2x^3} \right) = \lim_{x \rightarrow \infty} (-1.5x^2) = -1.5 \lim_{x \rightarrow \infty} (x^2) = -\infty.$$

Similarly, we find that

$$\lim_{x \rightarrow -\infty} g(x) = \frac{-3}{2} \lim_{x \rightarrow -\infty} (x^2) = -\infty.$$

## Exercise 3.3

Consider the rational function  $\ell(x) = \frac{2x^7 - 3x^3 + 1}{10x^8(3-2x)(5x-1)}$  and determine the following quantities:

- 1  $\lim_{x \rightarrow \infty} \ell(x)$ ;
- 2  $\lim_{x \rightarrow -\infty} \ell(x)$ ;
- 3  $\lim_{x \rightarrow 1} \ell(x)$ .

## Solution

- The leading term of the numerator is  $2x^7$ ;
- the leading term of the denominator is  $-100x^{10} = 10x^8 \cdot (-2x) \cdot 5x$ .

Thus, we obtain

$$\lim_{x \rightarrow \infty} \ell(x) = \lim_{x \rightarrow \infty} \left( \frac{2x^7}{-100x^{10}} \right) = \lim_{x \rightarrow \infty} \left( \frac{1}{-50x^3} \right) = 0.$$

Similarly,

$$\lim_{x \rightarrow -\infty} \ell(x) = \lim_{x \rightarrow -\infty} \left( \frac{1}{-50x^3} \right) = 0.$$

As  $x = 1$  is not a zero of the denominator, the function  $\ell(x)$  is continuous at  $x = 1$ . Therefore, we evaluate the limit directly:

$$\lim_{x \rightarrow 1} \ell(x) = \ell(1) = \frac{2 \cdot 1^7 - 3 \cdot 1^3 + 1}{10 \cdot 1^8 (3 - 2 \cdot 1)(5 \cdot 1 - 1)} = \frac{2 - 3 + 1}{10 \cdot 1 \cdot 1 \cdot 4} = \frac{0}{40} = 0.$$

# Rational Functions at Infinity: Leading Term Approximation

As we have observed, the behavior of a rational function

$$f(x) = \frac{p(x)}{q(x)}$$

as  $x \rightarrow \pm\infty$ , is determined by the leading terms of the numerator and denominator:

$$\lim_{x \rightarrow \pm\infty} f(x) = \lim_{x \rightarrow \pm\infty} \frac{a_n x^n}{b_m x^m} = \lim_{x \rightarrow \pm\infty} \alpha x^d,$$

where  $d = n - m$  is the difference of degrees and  $\alpha = \frac{a_n}{b_m}$  is the ratio of leading coefficients.

- If  $d > 0$ : the limit is  $\infty$  or  $-\infty$ , depending on the sign of  $\alpha$  and parity of  $d$ .
- If  $d = 0$ : the limit is  $\alpha$ .
- If  $d < 0$ : the limit is 0.

# Examples

Here are some examples illustrating the use of leading terms to compute limits:

- ①  $f(x) = \frac{3x^5+x^2-7}{x^5+4x+1} \Rightarrow \lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \left( \frac{3x^5}{x^5} \right) = \frac{3}{1} = 3;$
- ②  $g(x) = \frac{4x^7+6x^2+9}{2x^3+x+10} \Rightarrow \lim_{x \rightarrow \infty} g(x) = \lim_{x \rightarrow \infty} \left( \frac{4x^7}{2x^3} \right) \lim_{x \rightarrow \infty} 2x^4 = \infty;$
- ③  $h(x) = \frac{5x^2-4x+1}{6x^8+2x^3+3} \Rightarrow \lim_{x \rightarrow \infty} h(x) = \lim_{x \rightarrow \infty} \left( \frac{5x^2}{6x^8} \right) = \lim_{x \rightarrow \infty} \left( \frac{5}{6x^6} \right) = 0.$

# Summary

The behavior of a rational function  $f(x) = \frac{p(x)}{q(x)}$  as  $x \rightarrow \pm\infty$  is determined by the interplay between the leading terms of the numerator  $p(x)$  and the denominator  $q(x)$ . The ratio of their leading coefficients is denoted by  $\alpha$ :

$\alpha$	$\deg(p)$ Vs $\deg(q)$	$\lim_{x \rightarrow \infty} (f(x))$	$\lim_{x \rightarrow -\infty} (f(x))$
any $\alpha$	$\deg(p) < \deg(q)$	0	0
any $\alpha$	$\deg(p) = \deg(q)$	$\alpha$	$\alpha$
$\alpha > 0$	$\deg(p) - \deg(q) > 0$ , even	$\infty$	$\infty$
$\alpha > 0$	$\deg(p) - \deg(q) > 0$ , odd	$\infty$	$-\infty$
$\alpha < 0$	$\deg(p) - \deg(q) > 0$ , even	$-\infty$	$-\infty$
$\alpha < 0$	$\deg(p) - \deg(q) > 0$ , odd	$-\infty$	$\infty$

# Limits Involving Exponential Functions

Rational functions are not the only functions where limits as  $x \rightarrow \infty$  (or  $x \rightarrow -\infty$ ) play a crucial role. Many important applications in biology, physics, and other fields involve exponential functions. One fundamental limit that we will use repeatedly is:

$$\lim_{x \rightarrow \infty} e^{-x} = 0.$$

# Examples

We will consider a few examples.

- $\lim_{x \rightarrow \infty} \left( \frac{-7}{5 - e^{-x}} \right)$ .

Since  $\lim_{x \rightarrow \infty} e^{-x} = 0$ , the denominator approaches  $5 - 0 = 5$ .

Hence,

$$\lim_{x \rightarrow \infty} \left( \frac{-7}{5 - e^{-x}} \right) = \frac{-7}{5}.$$

- $\lim_{x \rightarrow \infty} \left( \frac{-3e^{5x}}{11e^{5x} - e^{2x}} \right)$ .

Dividing both numerator and denominator by  $e^{5x}$  gives

$$\lim_{x \rightarrow \infty} \left( \frac{-3e^{5x}}{11e^{5x} - e^{2x}} \right) = \lim_{x \rightarrow \infty} \left( \frac{-3}{11 - e^{-3x}} \right).$$

Since  $\lim_{x \rightarrow \infty} e^{-3x} = 0$ , the denominator approaches  $11 - 0 = 11$ , so

$$\lim_{x \rightarrow \infty} \left( \frac{-3}{11 - e^{-3x}} \right) = \frac{-3}{11}.$$