Lecture 22

Inverse trigonometric identities

MATH 0200

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Outline

Trigonometric functions composed with their inverses

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Example

Evaluate $\arccos(\cos(400^\circ))$.
First we need to find $0 \leq \alpha \leq \pi$ (or 180°) with $\cos(\alpha) = \cos(400^\circ)$. As $\cos(400^\circ) = \cos((400-360)^\circ) =$
$= \cos(40^\circ)$, we get $\alpha = 40^\circ = \frac{80\pi}{360} = \frac{2\pi}{9}$ and
$\arccos(\cos(400^\circ)) = \arccos\left(\cos\left(\frac{2\pi}{9}\right)\right) = \frac{2\pi}{9}$.

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3. $f^{-1}(x) = \frac{1}{3}\arcsin\left(\frac{5-x}{6}\right)$.

Check: $f \circ f^{-1}(x) = 5 - 6\sin(3 \cdot \frac{1}{3}\arcsin\left(\frac{5-x}{6}\right)) = 5 - 6\sin(\arcsin\left(\frac{5-x}{6}\right)) = 5 - 6 \cdot \left(\frac{5-x}{6}\right) = 5 - (5 - x) = x$ ✓

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Range of f^{-1} = domain of $f = \left[0, \frac{\pi}{6}\right]$.
As $\sin(3x)$ attains all values between $0 = \sin(0)$ and $1 = \sin\left(\frac{3\pi}{6}\right)$, we get domain of f^{-1} = range of f is $[-1, 5]$.

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2. Evaluate $\tan(\arccos(0.4))$.
   $\tan(\arccos(0.4)) = \frac{\sin(\arccos(0.4))}{\cos(\arccos(0.4))} = \frac{\sqrt{0.84}}{0.4} \approx 2.291$.

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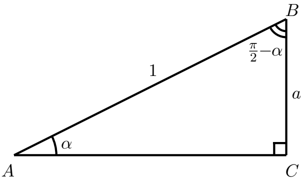

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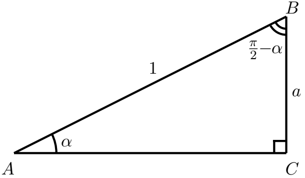

As $\sin(\alpha) = \cos\left(\frac{\pi}{2} - \alpha\right) = a/1 = a$, we get

$$\arcsin(a) + \arccos(a) = \alpha + \left(\frac{\pi}{2} - \alpha\right) = \frac{\pi}{2}.$$

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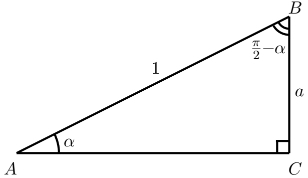

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The equation $\arcsin(a) + \arccos(a) = \frac{\pi}{2}$ holds true for any $-1 \leq a \leq 1$.

Question

Evaluate $\arcsin\left(\cos\left(\frac{\pi}{2} - 1\right)\right)$.