MATH 007 A

Lecture 2

Limits: How to Compute Them?

This Week's Assignments

- **Homework 3.1-3.3:** Due on *Monday* 10/06, 11:59 PM.

Outline

Limit Notation

In mathematics, when describing the behavior of a function as the independent variable approaches a particular value or infinity, it is common to use limit notation. Instead of writing "As x approaches a^+, $f(x)$ approaches something," we use

$$\lim_{x \to a^+} f(x) = \text{something},$$

where "something" can be a number, ∞, or $-\infty$. Similarly, for approaching from the left, we use

$$\lim_{x \to a^-} f(x).$$

When considering the behavior of a function as x tends towards infinity, we write

$$\lim_{x \to \infty} f(x) = \text{something},$$

and similarly for $-\infty$.

Example

Example

For the function $f(x) = \dfrac{75x - 22}{3(5-x)(2x-3)(x+7)}$, we have computed that it approaches ∞ as x approaches -7 from the right and $-\infty$ as x approaches -7 from the left.

Using limit notation, this can be expressed as

$$\lim_{x \to -7^+} f(x) = \infty,$$

$$\lim_{x \to -7^-} f(x) = -\infty.$$

Exercise 2.1

Exercise

Consider the rational function $g(x) = \frac{3-2x}{5(3-x)(x+4)}$.

1. Compute $\lim_{x \to -4^+} g(x)$.
2. Compute $\lim_{x \to 3^-} g(x)$.

Solution

Consider the rational function $g(x) = \frac{3-2x}{5(3-x)(x+4)}$.

$$\lim_{x\to -4^+} g(x) = \lim_{x\to -4^+} \frac{3-2\cdot(-4)}{5(3-(-4))\cdot(x+4)} = \lim_{x\to -4^+} \frac{3+8}{5(3+4)(x+4)}$$
$$= \lim_{x\to -4^+} \frac{11}{5\cdot 7\cdot (x+4)} = \lim_{x\to -4^+} \frac{11}{35}\cdot\frac{1}{x+4}.$$

Since $x+4>0$ for $x\to -4^+$ gives $\lim\limits_{x\to -4^+} \frac{1}{x+4} = \infty$ and $\frac{11}{35}>0$, we conclude $\lim\limits_{x\to -4^+} g(x) = \infty$.

$$\lim_{x\to 3^-} g(x) = \lim_{x\to 3^-} \frac{3-2\cdot 3}{5(3-x)(3+4)} = \lim_{x\to 3^-} \frac{3-6}{5(3-x)\cdot 7} = \lim_{x\to 3^-} \frac{-3}{35(3-x)}.$$

Since $3-x>0$ for $x\to 3^-$ gives $\lim\limits_{x\to 3^-} \frac{1}{3-x} = \infty$ and $-\frac{3}{35}<0$, we get $\lim\limits_{x\to 3^-} g(x) = -\infty$.

Basic Properties of Limits

Let $f(x)$ and $g(x)$ be two functions with $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$.

1. $\lim_{x \to a} \big(f(x) + g(x)\big) = L + M$ (Sum Rule)
2. $\lim_{x \to a} \big(f(x) - g(x)\big) = L - M$ (Difference Rule)
3. $\lim_{x \to a} \big(f(x) \cdot g(x)\big) = L \cdot M$ (Product Rule)
4. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$, if $M \neq 0$ (Quotient Rule)
5. $\lim_{x \to a} cf(x) = cL$, for any constant c (Constant Multiple Rule)

These properties allow us to evaluate limits directly for many functions.

Examples

Example

Compute $\lim_{x \to -2} \left(0.5x^3 - 7x + 4\right)$

Solution: using the rules 1 and 5 from the previous slide, we get

$$\lim_{x \to -2} \left(0.5x^3 - 7x + 4\right) = 0.5 \lim_{x \to -2} (x^3) - 7 \lim_{x \to -2} (x) + \lim_{x \to -2} (4) =$$
$$0.5 \cdot (-8) - 7 \cdot (-2) + 4 = 14.$$

We also notice that we could directly find
$f(-2) = 0.5 \cdot (-2)^3 - 7 \cdot (-2) + 4 = 14.$
More generally, it is true that if a function $f(x)$ is *continuous* at a point $x = a$, then by definition, $\lim_{x \to a} f(x) = f(a)$. Since polynomials are continuous everywhere, we could have directly computed $f(-2)$.

Examples

Example

Compute $\lim_{x\to 0} \frac{1-3x^5}{2x^2+27}$.

Solution: since $2x^2+27$ at $x=0$ is $2\cdot 0^2+27=27\neq 0$, we can use the quotient rule:

$$\lim_{x\to 0} \frac{1-3x^5}{2x^2+27} = \frac{\lim_{x\to 0}(1-3x^5)}{\lim_{x\to 0}(2x^2+27)} = \frac{1-3\cdot 0^5}{27} = \frac{1}{27}.$$

Exercise 2.2

Exercise

Compute $\lim_{x \to -1} (7 - 3x^2 + 2x^3)$.

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Compute $\lim_{x \to -1} (7 - 3x^2 + 2x^3)$.

Solution

Exercise

Compute $\lim_{x \to -1} (7 - 3x^2 + 2x^3)$.

Using properties of limits, we find

$$\lim_{x \to -1} (7 - 3x^2 + 2x^3) = \lim_{x \to -1} (7) - 3 \lim_{x \to -1} (x^2) + 2 \lim_{x \to -1} (x^3) =$$
$$7 - 3 \cdot (-1)^2 + 2 \cdot (-1)^3 = 7 - 3 - 2 = 2.$$