MATH 007 A

Lecture 18

How to Sketch a Graph of a Function?

This Week's Assignments

- **Homework 5.5, 5.6:** Due on *Friday* 12/05, 11:59 PM.
- **Microtutorials 9 and 10:** Due on *Friday* 12/05, 11:59 PM.

Gathering Information About a Function

Today we will learn how to use analytical information about a function in order to produce an accurate sketch of its graph.

Recall the following key ideas:

1. the first derivative, $f'(x)$, allows to determine intervals where the function is *increasing* or *decreasing*;
2. second derivative, $f''(x)$, allows to determine intervals where the function is *concave up* or *concave down*.

Using these, we can locate:

- **critical points** (where $f'(c) = 0$ or $f'(c)$ is undefined);
- **local max/min**, via the First or Second Derivative Test;
- **inflection points**, where the concavity changes.

Goal: use all of this information to create a reliable sketch of the graph that reflects the function's key features.

Gathering Information About a Function

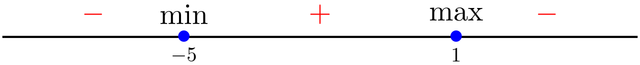
In addition, to refine the sketch, we also determine:

- **asymptotic behavior** of $f(x)$ as $x \to \pm\infty$,
- x**- and** y**-intercepts**.

Example: $f(x) = 3 + 5x - 2x^2 - \frac{x^3}{3}$

We start by computing the first derivative:
$f'(x) = 5 - 4x - x^2 = (x+5)(1-x)$, so the critical points are $x = -5$ and $x = 1$.

To determine where the function is increasing or decreasing, we examine the sign of $f'(x)$:

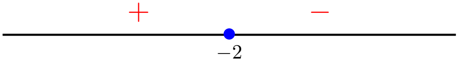


From the sign chart, we conclude that $f(x)$ is **decreasing** on $(-\infty, -5) \cup (1, \infty)$, **increasing** on $(-5, 1)$, and **decreasing** on $(1, \infty)$. Thus, $x = -5$ corresponds to a local minimum and $x = 1$ corresponds to a local maximum.

Example: $f(x) = 3 + 5x - 2x^2 - \frac{x^3}{3}$

Next, compute the second derivative: $f''(x) = -4 - 2x$. Since $f''(-5) = 6 > 0$, the point $(-5, f(-5))$ is a local minimum; and since $f''(1) = -6 < 0$, the point $(1, f(1))$ is a local maximum.

Finally, solving $f''(x) = 0$ gives $x = -2$. Because the concavity changes there, $(-2, f(-2))$ is the (only) inflection point:



From the sign of $f''(x)$, we conclude that the function is **concave up** on $(-\infty, -2)$ and **concave down** on $(-2, \infty)$.

Example: $f(x) = 3 + 5x - 2x^2 - \frac{x^3}{3}$

Next, we evaluate the function at the important points:

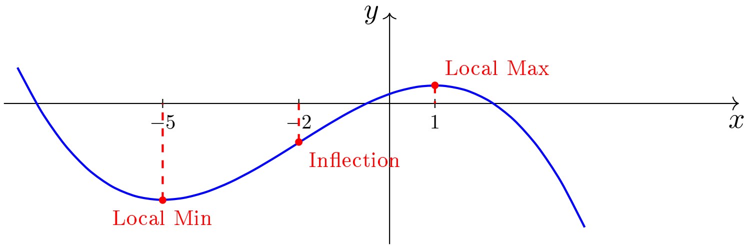
- $f(-5) = 3 + 5 \cdot (-5) - 2 \cdot (-5)^2 - \dfrac{(-5)^3}{3} = -30\tfrac{1}{3};$
- $f(1) = 3 + 5 \cdot 1 - 2 \cdot 1^2 - \dfrac{1}{3} = 5\tfrac{2}{3};$
- $f(-2) = 3 + 5 \cdot (-2) - 2 \cdot (-2)^2 - \dfrac{(-2)^3}{3} = -12\tfrac{1}{3}.$

and analyze the behavior of $f(x)$ as $x \to \pm\infty$:

- $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} \left(3 + 5x - 2x^2 - \frac{x^3}{3}\right) = -\frac{1}{3} \cdot \lim\limits_{x \to -\infty} \left(x^3\right) = \infty;$
- $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} \left(3 + 5x - 2x^2 - \frac{x^3}{3}\right) = -\frac{1}{3} \cdot \lim\limits_{x \to -\infty} \left(x^3\right) = \infty.$

Example: $f(x) = 3 + 5x - 2x^2 - \frac{x^3}{3}$

Combining the data from the previous three slides, we obtain the following sketch of the graph:



One More Example: $g(x) = e^{-x^2/2}$

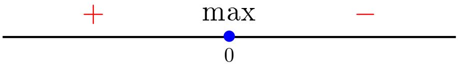
Let's carry out a similar analysis for the function $g(x) = e^{-x^2/2}$.
The first derivative is

$$g'(x) = e^{-x^2/2} \cdot (-x) = -x\, e^{-x^2/2}.$$

The derivative equals zero only at $x = 0$, so this is the only critical point.
As $e^{-x^2/2} > 0$ for all x, the sign of $g'(x)$ coincides with that of $-x$:

$$\begin{cases} g'(x) > 0 & \text{if } x < 0, \\ g'(x) < 0 & \text{if } x > 0. \end{cases}$$

Thus, the function increases for $x < 0$ and decreases for $x > 0$, so the point $(0, 1)$ is a local maximum.



One More Example: $g(x) = e^{-x^2/2}$

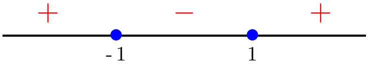
Next, compute the second derivative (using the chain rule):

$$g''(x) = \left(-xe^{-x^2/2}\right)' = -e^{-x^2/2} + x^2 e^{-x^2/2} = (x^2 - 1)e^{-x^2/2}.$$

Since the $e^{-x^2/2}$ is always positive, the sign of $g''(x)$ depends only on $x^2 - 1 = (x+1)(x-1)$:

$$g''(x) \begin{cases} < 0, & -1 < x < 1, \\ > 0, & x < -1 \text{ or } x > 1. \end{cases}$$

Thus the concavity changes at the solutions to $g''(x) = 0$, namely $x = \pm 1$. These are the two inflection points:



One More Example: $g(x) = e^{-x^2/2}$

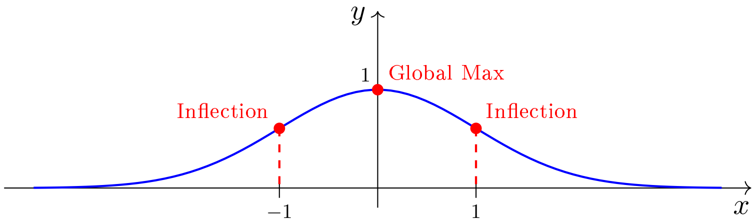
For completeness, we evaluate the function at critical and inflection points:

$$g(0) = 1, \qquad g(1) = e^{-1/2}, \qquad g(-1) = e^{-1/2};$$

analyze the behavior as $x \to \pm\infty$:

$$\lim_{x \to \pm\infty} g(x) = \lim_{x \to \pm\infty} e^{-x^2/2} = 0.$$

Combining this information, we obtain the following sketch:



Gauss in the Haus

The curve we just analyzed, $g(x) = e^{-x^2/2}$, is not only a convenient example-it is the graph of the *normal* (or *Gaussian*) distribution, named after the famous German mathematician **Carl Friedrich Gauss**, one of the most influential in history.

- Many small, independent effects acting together tend to produce outcomes whose overall pattern is very close to this distribution.
- This phenomenon lies behind the *Central Limit Theorem*: the average distribution of many such independent contributions, resembles the Gaussian.
- Because of this universality, a huge portion of modern statistics, including the most widely used hypothesis tests, are built on the assumption that real-world data is distributed approximately in the shape of this curve.