

MATH *007*^F A

Lecture 17

L'Hôpital's Rule

This Week's Assignments

- **Homework 5.3:** Extended until *Wednesday* 11/26, 11:59 PM.
- **Microtutorials 7 and 8:** Due on *Wednesday* 11/26, 11:59 PM.
- **Quiz 5 (last one!):** complete during your group's Discussion session.

L'Hôpital's Rule

Oftentimes (might be a slight exaggeration 😊), we encounter limits of the form

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)},$$

where both

$$\lim_{x \rightarrow x_0} f(x) = 0 = \lim_{x \rightarrow x_0} g(x) \text{ or } \lim_{x \rightarrow x_0} f(x) = \pm\infty, \quad \lim_{x \rightarrow x_0} g(x) = \pm\infty,$$

known, respectively, as the “0/0” and “ ∞/∞ ” indeterminate forms. Then, we may compute the limit using L'Hôpital's Rule:

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}, \quad \text{provided the latter limit exists.}$$

Remark

The rule is also applicable when $x_0 = \infty$ or $x_0 = -\infty$.

Examples

Example

Let's use L'Hôpital's Rule to evaluate the following limits.

$$\textcircled{1} \lim_{x \rightarrow 5} \frac{25 - x^2}{x - 5}.$$

We begin by computing the separate limits:

$$\lim_{x \rightarrow 5} (25 - x^2) = 0 \text{ and } \lim_{x \rightarrow 5} (x - 5) = 0.$$

Thus, this is a $0/0$ indeterminate form. L'Hôpital's Rule gives:

$$\lim_{x \rightarrow 5} \frac{25 - x^2}{x - 5} = \lim_{x \rightarrow 5} \frac{(25 - x^2)'}{(x - 5)'} = \lim_{x \rightarrow 5} \frac{-2x}{1} = -10.$$

Remark

Notice that we could also compute the limit by factoring $25 - x^2 = (5 - x)(5 + x)$, giving the same result:

$$\lim_{x \rightarrow 5} \frac{25 - x^2}{x - 5} = \lim_{x \rightarrow 5} \frac{(5 - x)(5 + x)}{x - 5} = \lim_{x \rightarrow 5} -(5 + x) = -(5 + 5) = -10.$$

Examples

Example

$$② \lim_{x \rightarrow 0} \frac{1-3^x}{2x}.$$

First we evaluate the respective limits in the numerator and denominator:

$$\lim_{x \rightarrow 0} (1 - 3^x) = 1 - 1 = 0, \text{ and } \lim_{x \rightarrow 0} (2x) = 0.$$

This is again a $0/0$ indeterminate form, so L'Hôpital's Rule applies and gives:

$$\lim_{x \rightarrow 0} \frac{1-3^x}{2x} = \lim_{x \rightarrow 0} \frac{(1-3^x)'}{(2x)'} = \lim_{x \rightarrow 0} \frac{-3^x \ln(3)}{2} = -\frac{\ln(3)}{2}.$$

Examples

Example

$$\textcircled{8} \quad \lim_{x \rightarrow \infty} \frac{x^3 - x}{e^x}.$$

First we check the numerator and denominator separately:

$$\lim_{x \rightarrow \infty} (x^3 - x) = \infty, \text{ and } \lim_{x \rightarrow \infty} e^x = \infty.$$

Thus we have an indeterminate form ∞/∞ , so L'Hôpital's Rule applies:

$$\lim_{x \rightarrow \infty} \frac{x^3 - x}{e^x} = \lim_{x \rightarrow \infty} \frac{(x^3 - x)'}{(e^x)'} = \lim_{x \rightarrow \infty} \frac{3x^2 - 1}{e^x}.$$

The resulting limit is also an indeterminate form ∞/∞ , so we apply L'Hôpital's Rule again:

$$\lim_{x \rightarrow \infty} \frac{3x^2 - 1}{e^x} = \lim_{x \rightarrow \infty} \frac{(3x^2 - 1)'}{(e^x)'} = \lim_{x \rightarrow \infty} \frac{6x}{e^x}.$$

Examples

Example

This is again ∞/∞ , so we apply L'Hôpital's Rule once more:

$$\lim_{x \rightarrow \infty} \frac{6x}{e^x} = \lim_{x \rightarrow \infty} \frac{(6x)'}{(e^x)'} = \lim_{x \rightarrow \infty} \frac{6}{e^x} = 0,$$

Remark

Let $f(x)$ be a polynomial of degree n . In Lecture 11 (on higher-order derivatives) we showed that the $(n + 1)$ st derivative, $f^{(n+1)}(x)$, is identically zero. In contrast, the exponential function is “immune” to differentiation: $(e^x)' = e^x$.

Therefore, when evaluating limits of the form $\lim_{x \rightarrow \infty} \frac{f(x)}{e^x}$, each application of L'Hôpital's Rule reduces the degree of the polynomial in the numerator. After $n + 1$ applications, the numerator becomes 0 while the denominator remains e^x . Hence, the limit is equal to 0.

Exercise 17.1

Exercise

Use L'Hôpital's Rule to compute the limit $\lim_{x \rightarrow 0} \frac{3-3 \cos(x)}{5x}$.

Solution

First check the form of the limit:

$$\lim_{x \rightarrow 0} (3 - 3 \cos(x)) = 3 - 3 \cdot \cos(0) = 3 - 3 \cdot 1 = 0 \text{ and } \lim_{x \rightarrow 0} 5x = 0.$$

Thus the expression is of the indeterminate form $0/0$, and L'Hôpital's Rule applies.

Compute the derivatives of the numerator and the denominator:

$$(3 - 3 \cos(x))' = 3 \sin(x) \text{ and } (5x)' = 5$$

and apply the rule to obtain

$$\lim_{x \rightarrow 0} \frac{3 - 3 \cos(x)}{5x} = \lim_{x \rightarrow 0} \frac{3 \sin(x)}{5} = \frac{3}{5} \cdot \lim_{x \rightarrow 0} \sin(x) = \frac{3}{5} \cdot 0 = 0.$$

Exercise 17.2

Exercise

Use L'Hôpital's Rule to compute the limit $\lim_{x \rightarrow 0} \frac{6\left(\ln(1+x) - x + \frac{x^2}{2}\right)}{x^3}$.

Solution

First check the form of the limit:

$$\lim_{x \rightarrow 0} 6 \left(\ln(1+x) - x + \frac{x^2}{2} \right) = 6(\ln(1) - 0 + 0) = 0 \text{ and } \lim_{x \rightarrow 0} x^3 = 0.$$

Thus the limit has the indeterminate form $0/0$, and L'Hôpital's Rule applies.

Apply L'Hôpital's Rule:

$$\lim_{x \rightarrow 0} \frac{6 \left(\ln(1+x) - x + \frac{x^2}{2} \right)}{x^3} = 6 \lim_{x \rightarrow 0} \frac{\frac{1}{1+x} - 1 + x}{3x^2}.$$

This is still $0/0$, so apply L'Hôpital's Rule again:

$$6 \lim_{x \rightarrow 0} \frac{\frac{1}{1+x} - 1 + x}{3x^2} = 6 \lim_{x \rightarrow 0} \frac{-\frac{1}{(1+x)^2} + 1}{6x}.$$

This is again a $0/0$ form, so apply L'Hôpital's Rule once more:

$$6 \lim_{x \rightarrow 0} \frac{-\frac{1}{(1+x)^2} + 1}{6x} = \lim_{x \rightarrow 0} \frac{2}{(1+x)^3} = 2.$$

One More Example

We have observed how L'Hôpital's Rule can be applied to limits of the types $0/0$ and ∞/∞ . However, some other limits can be rewritten to fit these forms and, hence, L'Hôpital's Rule becomes applicable.

Example

Evaluate the limit $\lim_{x \rightarrow 0^+} x^4 \ln(x)$.

We start by noticing that $\lim_{x \rightarrow 0^+} x^4 = 0$, while $\lim_{x \rightarrow 0^+} \ln(x) = -\infty$. So, it might seem that L'Hôpital's Rule is inapplicable. However, this is not true. If we write

$$x^4 \ln(x) = \frac{\ln(x)}{1/x^4},$$

then the limit becomes of the indeterminate form $-\infty/\infty$, and applying L'Hôpital's Rule gives

$$\lim_{x \rightarrow 0^+} \frac{\ln(x)}{1/x^4} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-4x^{-5}} = \lim_{x \rightarrow 0^+} \left(-\frac{x^4}{4} \right) = 0.$$

Optional Example

The example below is **optional** (beyond the modest goals of this course). Its purpose is to demonstrate how to handle limits of indeterminate form 1^∞ using logarithms and L'Hôpital's Rule.

Example

Evaluate the limit $\lim_{x \rightarrow 0} (1 - 5x)^{3/(x^2 - 4x)}$.

Step 1. Identify the indeterminate form. As $x \rightarrow 0$,

$$1 - 5x \rightarrow 1, \quad \frac{3}{x^2 - 4x} \rightarrow \infty,$$

so the expression has the indeterminate form 1^∞ .

Step 2. Take logarithms. Let $p = \lim_{x \rightarrow 0} (1 - 5x)^{3/(x^2 - 4x)}$. Then, assuming the limit exists,

$$\ln(p) = \lim_{x \rightarrow 0} \ln\left((1 - 5x)^{3/(x^2 - 4x)}\right) = \lim_{x \rightarrow 0} \frac{3\ln(1 - 5x)}{x^2 - 4x}.$$

Optional Example

Example

Step 3. Apply L'Hôpital's Rule. As $x \rightarrow 0$, the expression above is an indeterminate of the form $0/0$:

$$\ln(1 - 5x) \rightarrow 0, \quad x^2 - 4x \rightarrow 0.$$

Differentiate numerator and denominator:

$$\frac{d}{dx}(3\ln(1 - 5x)) = \frac{-15}{1-5x}, \quad \frac{d}{dx}(x^2 - 4x) = 2x - 4.$$

Thus, we find $\ln(p) = \lim_{x \rightarrow 0} \frac{-15/(1-5x)}{2x-4} = \frac{-15}{1} \cdot \frac{1}{-4} = \frac{15}{4}$.

Step 4. Exponentiate. We obtain $p = e^{15/4}$.