MATH 007 A

Lecture 16

Extrema and Inflection Points

This Week's Assignments

- **Homework 5.2:** Due on *Monday* 11/24, 11:59 PM.
- **Homework 5.3:** Extended until *Wednesday* 11/26, 11:59 PM.
- **Microtutorials 7 and 8:** Due on *Wednesday* 11/26, 11:59 PM.
- **Quiz 5 (last one!):** complete during your group's Discussion session.

Outline

Finding the Extrema: Two Tests

Let's start by recalling how we locate the points of local maxima and minima of a function $f(x)$. The First Derivative Test:

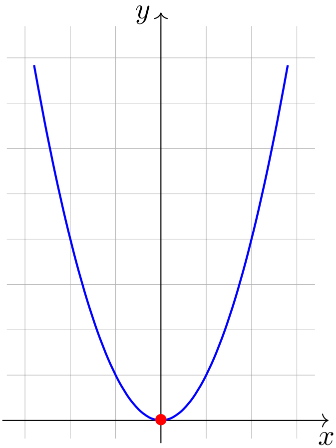
1. Find the critical points ($f'(c) = 0$);
2. For each such point, check how the sign of $f'(x)$ behaves when we pass through c:
   if f' changes from $-$ to $+$, then c is a local min;
   if f' changes from $+$ to $-$, then c is a local max;
   otherwise, no extremum at c.

Another way is to use the Second Derivative Test. At a critical point c:

- $f''(c) > 0 \Rightarrow$ local minimum.
- $f''(c) < 0 \Rightarrow$ local maximum.
- $f''(c) = 0 \Rightarrow$ the test is inconclusive.

We briefly mentioned this in Lecture 11 (see slide 12 therein). Let's take a look at some examples...

Example: $f(x) = x^2$

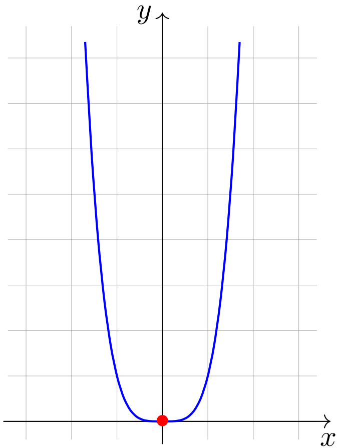


We compute the first derivative: $f'(x) = 2x$, so the only critical point is $x = 0$.

Second derivative: $f''(x) = 2$, hence $f''(0) = 2 > 0$. Therefore, $(0, 0)$ is a local (and in this case global) minimum.

Alternatively, we observe that $f'(x) = 2x$ changes sign from $-$ to $+$ as we pass through 0, which confirms the conclusion.

Example: $f(x) = x^4$



We compute $f'(x) = 4x^3$. Again, the unique critical point is $x = 0$.

The second derivative is $f''(x) = 12x^2$, so $f''(0) = 0$. The Second Derivative Test is inconclusive here.

However, $f'(x) = 4x^3$ changes sign from $-$ to $+$ as we pass through 0, so by the First Derivative Test, $(0, 0)$ is a local (and again global) minimum.

One More Example

Example

Consider the function $g(x) = \cos(\pi x^2)$ on the interval $[-1.1, 1.1]$. We use the chain rule to compute the first derivative:

$$g'(x) = (\cos(\pi x^2))' = -\sin(\pi x^2) \cdot (\pi x^2)' = -\sin(\pi x^2) \cdot 2\pi x.$$

So the critical points occur at the points where $g'(x) = 0$. This happens when either $\sin(\pi x^2) = 0$ or $x = 0$. On our domain, with $x^2 \leq 1.21$, the solutions to

$$\sin(\pi x^2) = 0 \quad \Longleftrightarrow \quad \pi x^2 = k\pi$$

are $x = \pm 1$ (corresponding to $k = 1$) and $x = 0$ (corresponding to $k = 0$).

One More Example

Example

Next we determine their types. Let's try the Second Derivative Test. Using the product rule and the fact that $(\sin(\pi x^2))' = \cos(\pi x^2) \cdot 2\pi x$, we compute the second derivative:

$$g''(x) = (-2\pi x \sin(\pi x^2))' = -2\pi \sin(\pi x^2) - 4\pi^2 x^2 \cos(\pi x^2).$$

Evaluating at the three critical points gives:

- $x = -1 : g''(-1) = -2\pi \sin(\pi) - 4\pi^2 \cdot (-1)^2 \cdot \cos(\pi) = 0 - 4\pi^2 \cdot (-1) = 4\pi^2 > 0$, so $x = -1$ is a local **minimum**.
- $x = 1 : g''(1) = -2\pi \sin(\pi) - 4\pi^2 \cdot 1^2 \cos(\pi) = 0 - 4\pi^2(-1) = 4\pi^2 > 0$, so $x = 1$ is also a local **minimum**.
- $x = 0 : g''(0) = -2\pi \sin(0) - 0 = 0$, so the Second Derivative Test is **inconclusive**.

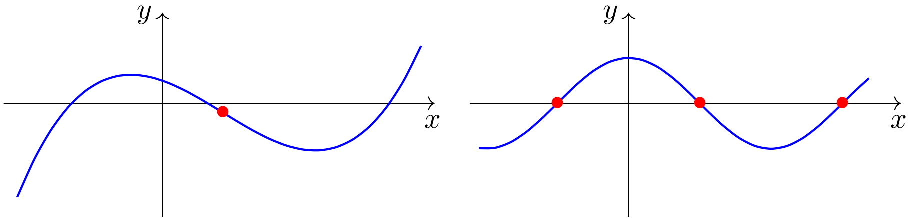
One More Example

Example

We can use the First Derivative Test to determine the type of the critical point at $x = 0$. However, we can also simply notice that $g(0) = \cos(0) = 1$, and since $\cos(\pi x^2) \leq 1$ for all x, we conclude that $g(x)$ has a global **maximum** at $x = 0$.

Inflection Points

Recall that an *inflection point* is a point on the curve where the concavity changes from up to down, or vice versa. Below are examples of inflection points on the graphs of a cubic function and of the cosine function.



Remark

As we can see, a function can have any number of inflection points, including none or infinitely many.

Exercise 16.1

Exercise

How many points of inflection can a function have? (Select all that apply.)

Answer. See the previous slide.

Inflection Points: How to Locate Them?

To locate the inflection points of a differentiable function, we find the points, satisfying the following:

- The second derivative vanishes, $f''(c) = 0$ or does not exist (see our example on slide 26 from the previous lecture).
- The second derivative changes sign at $x = c$.

Remark

Notice that the second condition is equivalent to the first derivative f' having a local maximum or minimum at $x = c$.

Examples

Example

Let's find the location of inflection points for the functions whose graphs we considered a few slides before, using their actual equations.

1. For the function $f(x) = x^3 - 5x^2 + 4$, we first compute

$$f'(x) = 3x^2 - 10x, \qquad f''(x) = 6x - 10.$$

   Setting $f''(x) = 0$ gives $6x - 10 = 0$, hence $x = \frac{5}{3}$.

   Checking the sign of $f''(x)$ around $x = \frac{5}{3}$ confirms a change of concavity. Therefore, $x = \frac{5}{3}$ is an inflection point.

Examples

Example

2. For the function $g(x) = \cos(x)$, we compute

$$g'(x) = -\sin(x), \qquad g''(x) = -\cos(x).$$

To locate possible inflection points, we solve $g''(x) = 0$:

$$-\cos(x) = 0 \quad \Longrightarrow \quad \cos(x) = 0.$$

This occurs whenever

$$x = \frac{\pi}{2} + k\pi, \qquad k \in \mathbb{Z}.$$

On each interval between consecutive such values, $\cos(x)$ changes sign, and therefore so does $g''(x)$. Hence the concavity of g switches at every point of the form $\frac{\pi}{2} + k\pi$.

Examples

On the picture below we show the graphs of $g(x) = \cos x$ (in blue) and its second derivative $g''(x) = -\cos x$ (in red, dashed). Notice that $g''(x)$ is obtained from $g(x)$ by a reflection across the x-axis. In particular, the x-intercepts are the same for both functions.
The regions where $g''(x)$ is positive or negative correspond exactly to intervals where $g(x)$ is concave up or concave down. At the points where $g''(x) = 0$, the concavity changes, giving inflection points: