MATH 007 A

Lecture 15

Monotonicity and Concavity

This Week's Assignments

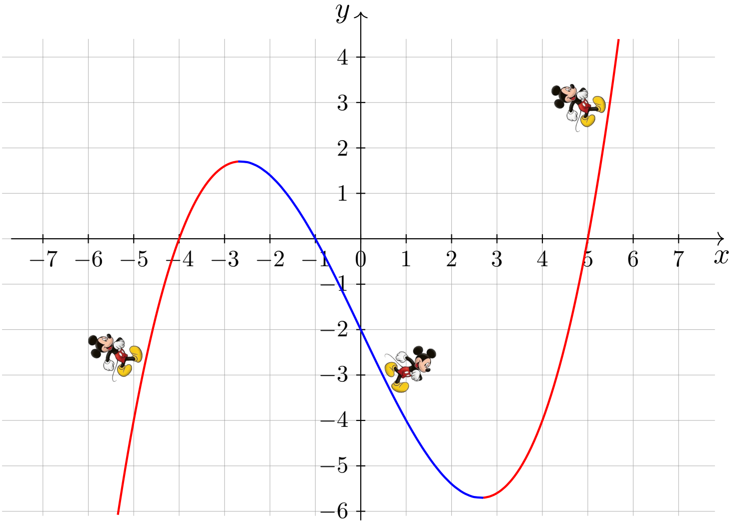
- **Homework 5.2, 5.3:** Due on *Monday* 11/24, 11:59 PM.
- **Microtutorials 7 and 8:** Due on *Wednesday* 11/26, 11:59 PM.

Outline

Functions: Increasing and Decreasing on Intervals

A function $f(x)$ is considered *increasing* on an interval (a, b) if, for any two values x_1 and x_2 in the interval such that $a < x_1 < x_2 < b$, the corresponding function values satisfy $f(x_1) < f(x_2)$.
In simpler terms, as we move from left to right along the interval, the function values consistently increase.
Conversely, a function $f(x)$ is termed *decreasing* on an interval $[a, b]$ if, for any two values x_1 and x_2 in the interval such that $a < x_1 < x_2 < b$, the corresponding function values satisfy $f(x_1) > f(x_2)$.
In this case, as we traverse the interval from left to right, the function values consistently decrease.
Graphically, an increasing function is represented by a rising curve on the specified interval, indicating that higher input values correspond to higher output values. Conversely, a decreasing function is represented by a descending curve, signifying that higher input values yield lower output values:

Functions: Increasing and Decreasing on Intervals



increasing: $(-\infty, -2.8) \cup (2.8, \infty)$ and decreasing: $(-2.8, 2.8)$

Monotonic Functions

A function is called *monotonic* (from Greek: *monos* = single, *tonos* = tone) if it always moves in the same direction: it is either always increasing or always decreasing on an interval.

More precisely:

- f is **increasing** on an interval I if $x_1 < x_2$ in I implies $f(x_1) \leq f(x_2)$.
- f is **decreasing** on an interval I if $x_1 < x_2$ in I implies $f(x_1) \geq f(x_2)$.

For instance, the function shown on the previous slide is *not* monotonic, because it switches between increasing and decreasing behavior.

First Derivative Test for Monotonicity

The Mean Value Theorem implies the following useful criterion:

First Derivative Test

Let $f(x)$ be differentiable on an interval I.

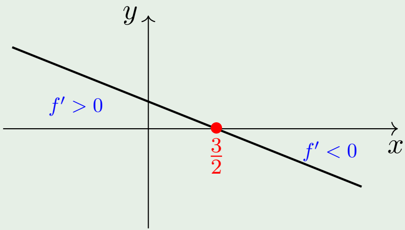
$$\text{If } f'(x) > 0 \text{ for all } x \in I, \text{ then } f \text{ is increasing on } I.$$
$$\text{If } f'(x) < 0 \text{ for all } x \in I, \text{ then } f \text{ is decreasing on } I.$$

First Derivative Test for Monotonicity

Example

Determine the intervals where the function $f(x) = 0.1(5-x)(2x+4)$ is increasing or decreasing.

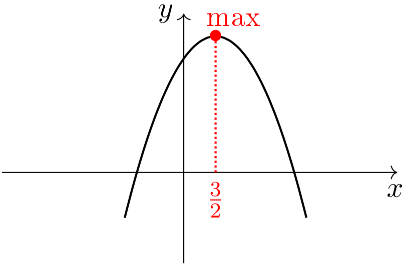
1. Compute the derivative using the product rule:
   $$f'(x) = 0.1((5-x)\cdot 2 + (2x+4)\cdot(-1)) = 0.1(10-2x-2x-4)$$
   $$= 0.6 - 0.4x.$$
2. Critical point: $f'(x) = 0 \Rightarrow x = \frac{3}{2}$.
3. Determine the sign of f':



First Derivative Test for Monotonicity

We conclude that

$$f'(x) > 0 \text{ when } x < \tfrac{3}{2} \quad \Rightarrow \quad f \text{ is increasing;}$$
$$f'(x) < 0 \text{ when } x > \tfrac{3}{2} \quad \Rightarrow \quad f \text{ is decreasing.}$$



First Derivative Test for Monotonicity

Example

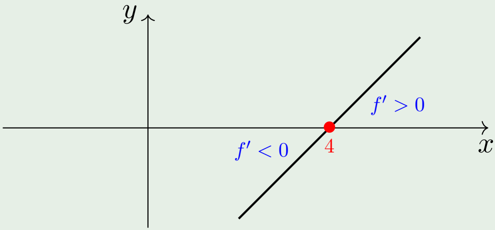
Determine the intervals where the function $g(x) = 2^{0.5x^2-4x}$ is increasing or decreasing.

1. Compute the derivative using the chain rule:
   $g'(x) = \ell n(2) \cdot 2^{0.5x^2-4x} \cdot (0.5x^2 - 4x)' = \ell n(2) \cdot 2^{0.5x^2-4x}\,(x-4).$
2. Find the critical point(s): since $2^{0.5x^2-4x} > 0$ for all x, we have that $g'(x) = 0$ occurs only when the linear factor is zero:
   $$x - 4 = 0 \quad \Rightarrow \quad x = 4.$$

First Derivative Test for Monotonicity

Example

3. Determine the sign of $g'(x)$: as $2^{0.5x^2-4x} > 0$ and $\ell n(2) > 0$, the sign of $g'(x)$ coincides with the sign of $x - 4$:



Conclusion:

$$\begin{cases} g(x) \text{ decreases on } (-\infty, 4), \\ g(x) \text{ increases on } (4, \infty). \end{cases}$$

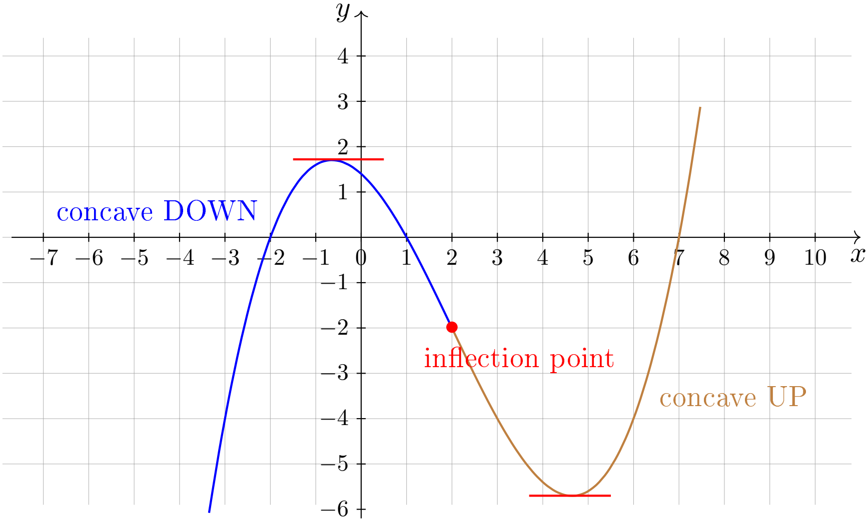
Concavity and Inflection Points

Imagine a function being like a hiking trail. When the trail forms an upward slope, creating a sort of terrain smile, we say the function is *concave up*. The graph lies above its tangent lines, resembling a hillside rising underfoot.

Conversely, when the trail dips downward, forming a terrain frown, the function is *concave down*. Here, the graph lies below its tangent lines, mirroring a descent into a valley.

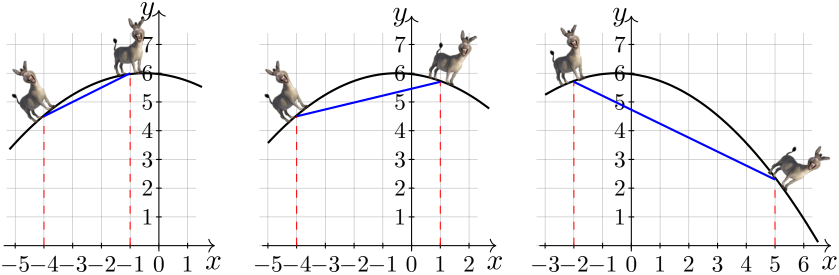
At certain points on our trail, the landscape changes dramatically. These points are like trail intersections where the concavity transforms—perhaps from a rising hillside to a descending valley. Such pivotal locations are known as *inflection points*.

Concavity and Inflection Point(s)



Adventures through the Prism of AROC

Now, consider two burros,  – , trekking along Riverside Mountains, with one following the other. The burros move straight without deviating left or right, and there is no backward movement or change in their order.
The planar graph of the boundary of our terrain represents the path of the burros. In this terrain analogy, when the burros climb a hill, corresponding to a concave-down function, the terrain at front burro's location is less steep than that at the back, leading to a decrease in AROC:



$\text{AROC}_1 = \frac{7-5.5}{-1-(-4)} = 0.5 \quad > \quad \text{AROC}_2 = \frac{6.8-5.5}{1-(-4)} = 0.26 \quad > \quad \text{AROC}_3 = \frac{3.3-6.8}{5-(-2)} = -0.5$

Remark

Conversely, when the terrain slopes downward into a valley (concave up function), the AROC increases.

Second Derivative Test for Concavity

Let $f(x)$ be ≥ 2 times differentiable on an interval I.

$$\text{If } f''(x) > 0 \text{ for all } x \in I, \text{ then } f \text{ is concave up on } I.$$
$$\text{If } f''(x) < 0 \text{ for all } x \in I, \text{ then } f \text{ is concave down on } I.$$

Example: Monotonicity and Concavity

Example

Consider the function $f(x) = \frac{x^3}{3} - x^2 - 8x + 5$.
We begin by computing its derivatives:

$$f'(x) = x^2 - 2x - 8, \qquad f''(x) = 2x - 2.$$

1. Critical Points and Monotonicity.
We find the critical points by solving $f'(c) = c^2 - 2c - 8 = 0$:

$$c^2 - 2c - 8 = 0 \quad \Longrightarrow \quad c = -2,\ 4.$$

A sign chart shows:

$$\begin{aligned} f'(x) &> 0 \quad \text{on } (-\infty, -2) \cup (4, \infty) \\ f'(x) &< 0 \quad \text{on } (-2, 4). \end{aligned}$$

Thus f increases on $(-\infty, -2)$ and $(4, \infty)$, and decreases on $(-2, 4)$.

Example: Monotonicity and Concavity

Example

2. Concavity and Inflection Point(s).

We check the sign of $f''(x)$:

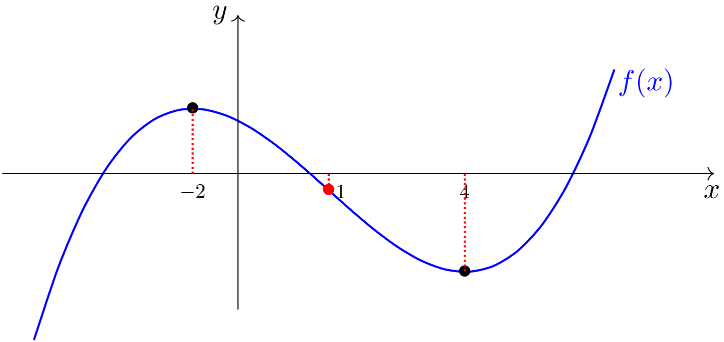
$$\begin{aligned} f''(x) < 0 \quad \text{for } x < 1 \quad &\Rightarrow \quad f \text{ is concave down,} \\ f''(x) > 0 \quad \text{for } x > 1 \quad &\Rightarrow \quad f \text{ is concave up.} \end{aligned}$$

As $f''(x)$ changes sign at $x = 1$, the point

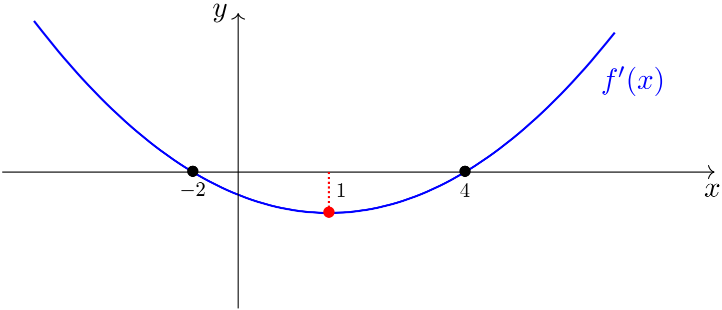
$$x = 1, \qquad f(1) = \frac{1}{3} - 1 - 8 + 1 = -\frac{23}{3},$$

is an *inflection point.*

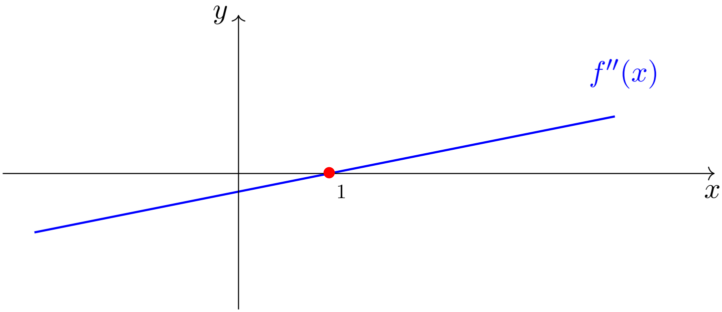
Example: Monotonicity and Concavity



Example: Monotonicity and Concavity



Example: Monotonicity and Concavity



Exercise 15.1

Exercise

Consider the function $g(x) = 5 + 6x - 1.5x^2 - x^3$.

(a) Determine the x-coordinate of the local maximum of g.

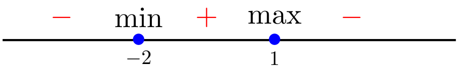
(b) Determine the x-coordinate of the local minimum of g.

(c) Determine the x-coordinate of the inflection point of g.

Solution

Step 1. First derivative and monotonicity.

$$g'(x) = 6 - 3x - 3x^2 = -3(x^2 + x - 2) = -3(x+2)(x-1):$$

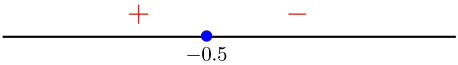


Critical points: $x = -2,\ x = 1.$

Thus, $g(x)$ is increasing on $(-2, 1)$ and decreasing elsewhere.

Step 2. Second derivative and concavity.

$$g''(x) = -3 - 6x = -3(1 + 2x):$$



Hence, $g(x)$ is concave up on $(-\infty, -0.5)$ and concave down on $(-0.5, \infty)$. Since $g''(x)$ changes sign at $x = -0.5$, it is an inflection point.

One More Example

Consider the function $g(x) = (7x - 2)^{1/3}$.
We compute the first derivative using the chain rule:

$$
\begin{aligned}
g'(x) &= \frac{1}{3}(7x-2)^{-2/3} \cdot (7x-2)' = \frac{1}{3}(7x-2)^{-2/3} \cdot 7 \\
&= \frac{7}{3}(7x-2)^{-2/3} = \frac{7}{3\sqrt[3]{(7x-2)^2}}.
\end{aligned}
$$

Observe that $g'(x)$ is *undefined* at $x = \frac{2}{7}$, since the cube root in the denominator becomes 0. For every other value of x, we have $g'(x) > 0$, because $(7x-2)^2$ is positive, and the cube root preserves sign. Therefore, $g(x)$ is increasing on each interval

$$(-\infty, \tfrac{2}{7}) \text{ and } (\tfrac{2}{7}, \infty).$$

As a continuous function cannot switch from increasing to decreasing at a single point, we conclude that $g(x)$ is increasing for all x.

One More Example

Concavity and Inflection Point(s).

Next we compute the second derivative (again using the chain rule):

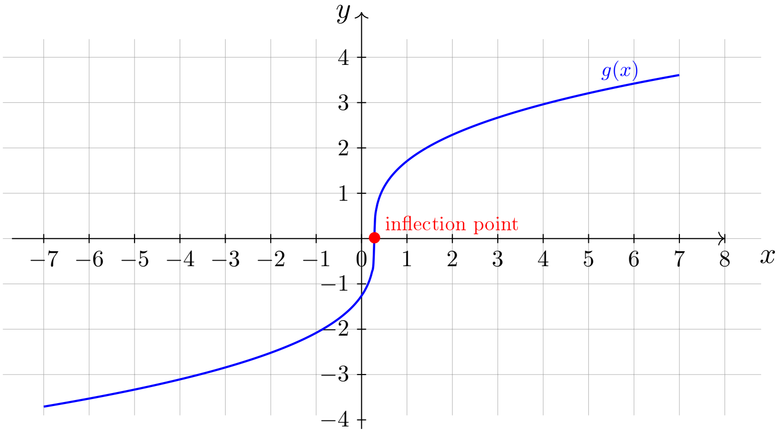
$$g''(x) = \left(\tfrac{7}{3}(7x-2)^{-2/3}\right)' = -\tfrac{2}{3} \cdot \tfrac{7}{3}(7x-2)^{-5/3} \cdot 7$$
$$= -\tfrac{98}{9}(7x-2)^{-5/3} = -\frac{98}{9\sqrt[3]{(7x-2)^5}}.$$

As the cube root and the fifth power preserve the sign, $\sqrt[3]{(7x-2)^5}$ has the same sign as $7x-2$. Since $-\frac{98}{9} < 0$, we conclude that

$$\begin{aligned} g''(x) > 0 \quad &\text{for } x < 2/7 \quad &\Rightarrow \quad &g \text{ is concave up,} \\ g''(x) < 0 \quad &\text{for } x > 2/7 \quad &\Rightarrow \quad &g \text{ is concave down.} \end{aligned}$$

As $g''(x)$ changes sign at $x = 2/7$, it is an *inflection point*.

One More Example



Last Example

Determine whether y increases or decreases with x on the branch of the hyperbola, given by $x^2 - y^2 = 36, x > 6, y > 0$.
We find $\frac{dy}{dx}$ by implicit differentiation:

$$2x - 2y \cdot \frac{dy}{dx} = 0 \Leftrightarrow 2y \cdot \frac{dy}{dx} = 2x \Leftrightarrow \frac{dy}{dx} = \frac{x}{y}.$$

As both $x > 6$ and $y > 0$, the ratio x/y is positive. Thus, y *is increasing as a function of* x on this region: