MATH *007* A
Lecture 14
Local Extrema and the Mean Value Theorem

This Week's Assignments

- **Homework 4.10, 5.1:** Due on *Wednesday* 11/19, 11:59 PM.
- **Microtutorial 6:** Due on *Wednesday* 11/19, 11:59 PM.
- **Quiz 4 (based on this lecture):** complete during your group's Discussion session.

Outline

Critical Points and Local Extrema

A **critical point** of a function $f(x)$ is a point $x = c$ where either:

- $f'(c) = 0$, or
- $f'(c)$ does not exist.

Local extrema (maximum or minimum points) can only occur at:

- Critical points, or
- Endpoints (if the domain is restricted).

Remark
Not every critical point is a local extremum!
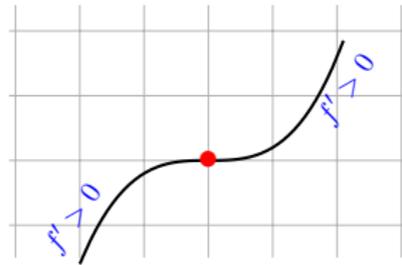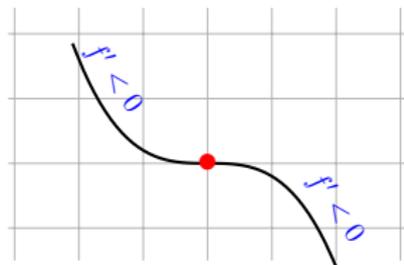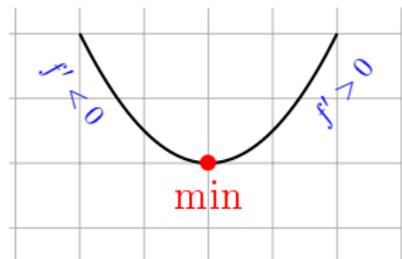
How to Determine Local Extrema?

Recall:
- If $f'(x) > 0$, then $f(x)$ is increasing.
- If $f'(x) < 0$, then $f(x)$ is decreasing.

First Derivative Test
- Look at the sign of $f'(x)$ on either side of a critical point (where $f'(x) = 0$ or undefined).
- If f' changes from $+$ to $-$, then f has a **local maximum**.
- If f' changes from $-$ to $+$, then f has a **local minimum**.
- If f' does not change sign, then f has **no extremum** at that point.
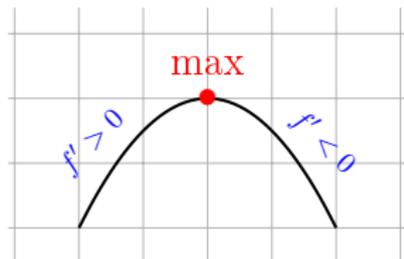
Example

Example

Let $f(x) = e^{3x - x^2}$.

(a) Find all c such that $f'(c) = 0$.

Use the chain rule to find the derivative:

$$f'(x) = e^{3x - x^2} \cdot (3x - x^2)' = e^{3x - x^2} \cdot (3 - 2x).$$

Setting this equal to 0:

$$f'(x) = 0 \quad \Leftrightarrow \quad e^{3x - x^2}(3 - 2x) = 0.$$

Since $e^{3x - x^2} > 0$ for all x, we must have:

$$3 - 2x = 0 \quad \Rightarrow \quad x = \frac{3}{2}.$$

Example: Critical Points and Local Extrema

Example

(b) Determine whether $f(x)$ has a local extremum at $x = \dfrac{3}{2} = 1.5$.

Use the first derivative test or observe the sign of $f'(x)$ near $x = 1.5$:

$$f'(x) = e^{3x - x^2}(3 - 2x).$$

For $x < 1.5$, we have $3 - 2x > 0 \Rightarrow f'(x) > 0$ (increasing).

For $x > 1.5$, we have $3 - 2x < 0 \Rightarrow f'(x) < 0$ (decreasing).

Conclusion: $f(x)$ has a local maximum at $x = 1.5$.

Global Max and Min on a Closed Interval

The Extreme Value Theorem

If a function $f(x)$ is **continuous** on a closed interval $[a, b]$, then $f(x)$ **attains a global maximum and a global minimum** on $[a, b]$.

**Finding global extrema of $f(x)$ on $[a, b]$**

1. Find all critical points in (a, b) where $f'(x) = 0$.
2. Evaluate f at each critical point and at the endpoints a and b.
3. The largest value is the global max; the smallest is the global min.

Example: Sketching Global and Local Extrema

Sketch a continuous function on $[-2, 5]$ that:
- has a **global maximum** at one of the endpoints,
- a **local minimum** at a point **inside** the interval,
- and a **local maximum** at a point **inside** the interval.
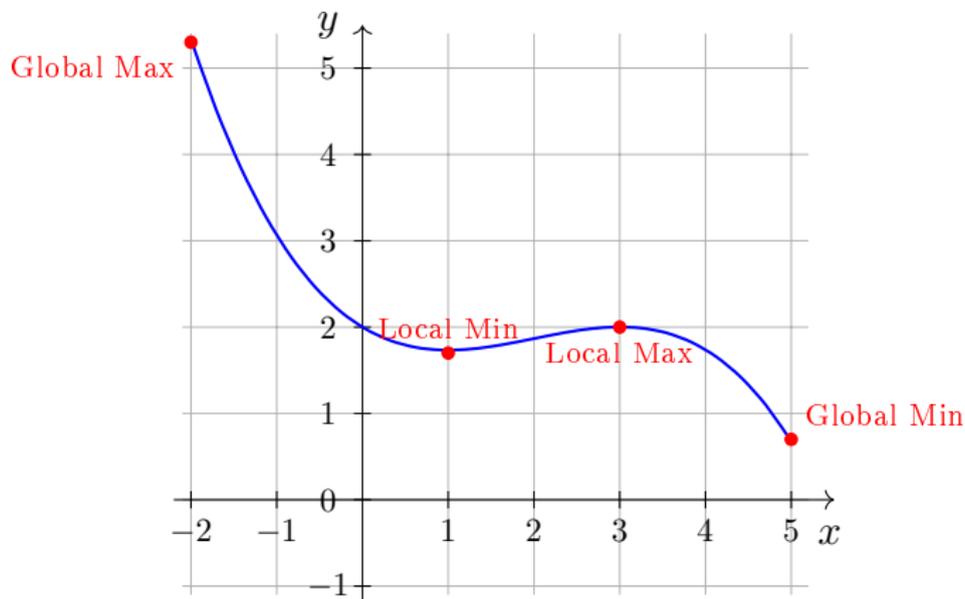
Example: Global Extrema on a Closed Interval

Find the global maximum and minimum of
$f(x) = 2x^3 + 3x^2 - 12x + 3$ on the interval $[-3, 2]$.

- Find critical points by solving $f'(x) = 0$:

  $f'(x) = 6x^2 + 6x - 12 = 6(x^2 + x - 2) = 6(x + 2)(x - 1) = 0.$

  Critical points: $x = -2, \ 1.$

- Evaluate $f(x)$ at $x = -3, \ -2, \ 1, \ 2$:

  $f(-3) = 2 \cdot (-27) + 3 \cdot 9 + 36 + 3 = -54 + 27 + 36 + 3 = 12$

  $f(-2) = 2 \cdot (-8) + 3 \cdot 4 + 24 + 3 = -16 + 12 + 24 + 3 = 23$

  $f(1) = 2 + 3 - 12 + 3 = -4$

  $f(2) = 16 + 12 - 24 + 3 = 7$

- **Conclusion:** global max at $x = -2$, value 23; global min at $x = 1$, value -4.

Mean Value Theorem

If $f(x)$ is a continuous function on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number $a < c < b$ with

$$f'(c) = \text{AROC} = \frac{f(b) - f(a)}{b - a}.$$

In other words, if the rockhopper penguin Cody catches the wave in the shape of a curve $f(x)$ at $(a, f(a))$ and stays on it until the point $(b, f(b))$, then inevitably at some point his surfing board will be parallel to the chord connecting $(a, f(a))$ and $(b, f(b))$.

Mean Value Theorem
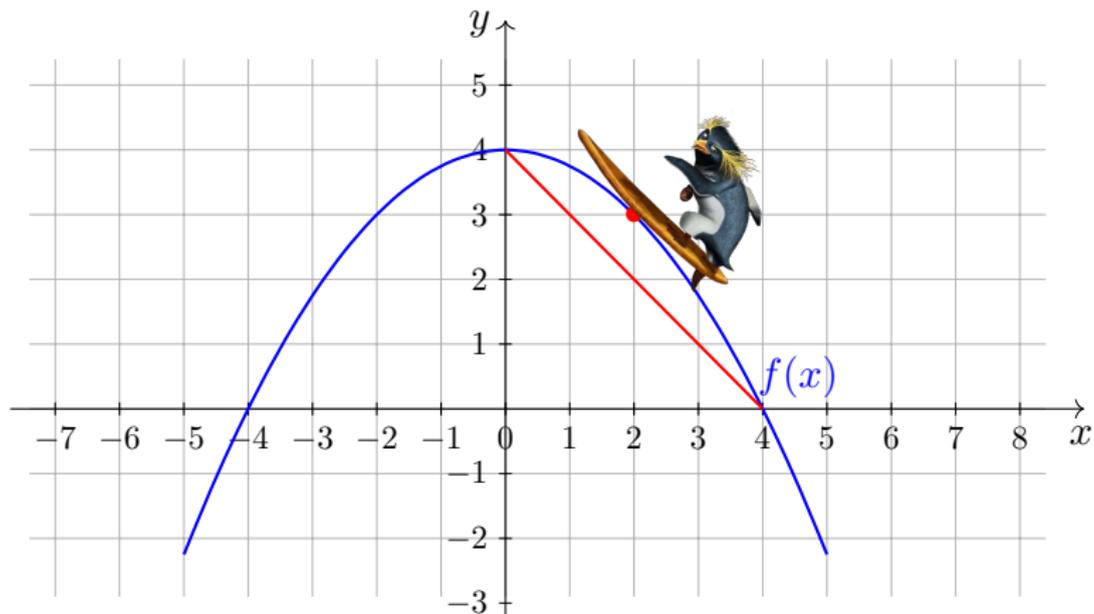
Example

Let $f(x) = x^3 - 7x + 5$ on the closed interval $[0, 1]$.

(a) What does the Mean Value Theorem assert about $f(x)$?

Since f is continuous on $[0, 1]$ and differentiable on $(0, 1)$, the MVT guarantees that there exists $0 < c < 1$ such that

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} = \frac{-1 - 5}{1} = -6.$$

Compute $f'(x) = 3x^2 - 7$, so the equation becomes:

$$3c^2 - 7 = -6 \Rightarrow 3c^2 = 1 \Rightarrow c^2 = \frac{1}{3}.$$

(b) Find all numbers c that satisfy the conclusion of the MVT.

Solving $c^2 = \frac{1}{3}$ gives $c = \pm\frac{1}{\sqrt{3}}$. Only the positive root lies in the interval $(0, 1)$ and satisfies the conclusion of the MVT.

Example

Suppose that $f(x) = \ell n(x)$ on the interval $[1, e]$.

(a) Find the slope of secant line connecting $(1, 0)$ and $(e, 1)$.

The slope of the secant line is
$$\frac{f(e) - f(1)}{e - 1} = \frac{\ell n(e) - \ell n(1)}{e - 1} = \frac{1 - 0}{e - 1} = \frac{1}{e - 1}.$$

(b) Explain why there exists a number $c \in (1, e)$ such that $f'(c)$ equals the slope from part (a), and find all such numbers.

Since $f(x) = \ell n(x)$ is continuous on $[1, e]$ and differentiable on $(1, e)$, the **Mean Value Theorem** asserts the existence of a number $c \in (1, e)$ with the required property.

Since $f'(x) = \frac{1}{x}$, we find c by solving
$$\frac{1}{c} = \frac{1}{e - 1} \implies c = e - 1.$$

Finally, because $1 < e - 1 < e$, the solution $c = e - 1$ indeed lies in the interval $(1, e)$.

Consider the function $f(x) = 2 - |2x - 1|$ on the interval $[0, 3]$.

(a) Find a formula for $f'(x)$. What is the domain of $f'(x)$?

$$f(x) = \begin{cases} 2 - (2x - 1) = 3 - 2x & \text{if } x \geq 0.5, \\ 2 - (-(2x - 1)) = 2x + 1 & \text{if } x < 0.5, \end{cases}$$

so the derivative is:

$$f'(x) = \begin{cases} 2 & \text{if } x < 0.5, \\ -2 & \text{if } x > 0.5. \end{cases}$$

The domain of $f'(x)$ does **not** include $x = 0.5$.

(b) Show that there is no $0 < c < 3$ such that $f'(c) = \dfrac{f(3) - f(0)}{3}$.

We compute:
$$\frac{f(3) - f(0)}{3} = \frac{-3 - 1}{3} = -\frac{4}{3},$$
but $f'(x)$ only takes values 2 and -2 on $(0, 3)$.

(c) Why does this not contradict the MVT?

The function f is not differentiable at $x = 0.5$, so the hypotheses of the Mean Value Theorem are not satisfied on $(0, 3)$.

Example: Bounding a Population Using the MVT

Let $N(t)$ denote the size of a population at time t. Suppose:

- $N(0) = 20$;
- $\left| \dfrac{dN}{dt} \right| \leq 15$ for all $t \in [0, 7]$.

What can we say about $N(7)$?

By the **Mean Value Theorem**, there exists some $c \in (0, 7)$ such that:

$$\frac{N(7) - N(0)}{7} = N'(c).$$

Taking absolute values and using the bound on the derivative:

$$\frac{N(7) - N(0)}{7} \leq 15 \Leftrightarrow -15 \leq \frac{N(7) - 20}{7} \leq 15$$

$$\Leftrightarrow -105 \leq N(7) - 20 \leq 105 \Leftrightarrow -85 \leq N(7) \leq 125.$$

However, since $N(t)$ represents a population, we must have $N(t) \geq 0$ for all t, so $0 \leq N(7) \leq 125$.

Exercise 15.1

Exercise

Let $f(x) = \sin(3x)$ on the interval $\left[\frac{\pi}{3}, \pi\right]$.

Use the Mean Value Theorem to find all values of c in $\left(\frac{\pi}{3}, \pi\right)$ such that

$$f'(c) = \frac{f(\pi) - f\left(\frac{\pi}{3}\right)}{\pi - \frac{\pi}{3}}.$$

Select all values of c that satisfy this equation.

Solution

We are given $f(x) = \sin(3x)$, so $f'(x) = 3\cos(3x)$. The average rate of change in $f(x)$ over the interval $\left[\frac{\pi}{3}, \pi\right]$ is

$$\frac{f(\pi) - f\left(\frac{\pi}{3}\right)}{\pi - \frac{\pi}{3}} = \frac{\sin(3\pi) - \sin(\pi)}{\frac{2\pi}{3}} = \frac{0 - 0}{\frac{2\pi}{3}} = 0.$$

Since $c \in \left(\frac{\pi}{3}, \pi\right)$, we have $3c \in (\pi, 3\pi)$. The values of $3c$ in this interval where $\cos(3c) = 0$ are

$$3c = \frac{3\pi}{2} \Leftrightarrow c = \frac{\pi}{2}, \text{ and } 3c = \frac{5\pi}{2} \Leftrightarrow c = \frac{5\pi}{6}.$$

Final answer: $c = \frac{\pi}{2}$ and $c = \frac{5\pi}{6}$.