MATH 007 A

Lecture 13

Derivatives of Inverse and Logarithmic Functions

This Week's Assignments

- **Homework 4.8, 4.9:** Due on *Monday* 11/10, 11:59 PM.
- **Microtutorial 5:** Due on *Monday* 11/10, 11:59 PM.

Outline

Exercise 13.1

Exercise

Let $f(x) = e^{\ell n(5x^3 - 7x^2 + x - 1)}$. Compute the fourth derivative $f^{(4)}(100)$, and enter your answer as a decimal.

Solution

First, simplify the function:

$$f(x) = e^{\ell n(5x^3 - 7x^2 + x - 1)} = 5x^3 - 7x^2 + x - 1,$$

since $e^{\ell n(g(x))} = g(x)$ when $g(x) > 0$.
Now, observe that $f(x)$ is a degree 3 polynomial. Therefore, its fourth derivative is zero:

$$f^{(4)}(x) = 0 \quad \text{for all } x.$$

Thus, $f^{(4)}(100) = 0$.

Remark

If we had tried to compute the fourth derivative directly from the original expression without simplifying, the calculation would be more complicated and error-prone. Recognizing the identity $e^{\ell n(g(x))} = g(x)$ is crucial for efficiency.

Inverse Functions: Motivating Example

Suppose you have a thermometer that only displays temperature in Fahrenheit, but you need to convert it back to Celsius. The temperature conversion formulas are as follows: for any temperature T in Celsius, the corresponding Fahrenheit temperature is given by $f(T) = 1.8T + 32$.

Question

If the thermometer reads 86°F, what is the temperature in Celsius?

To find $f^{-1}(86)$, we solve for T in the equation $1.8T + 32 = 86$. The result shows how to "undo" the original function f and motivates the need for an inverse function:

$$1.8T + 32 = 86 \Leftrightarrow 1.8T = 86 - 32 = 54 \Leftrightarrow T = \frac{54}{1.8} = 30^\circ C.$$

Inverse Processes in Biology

- In biology, certain processes have natural "inverse" or "reverse" processes that can restore the original state or perform the opposite function.
- Understanding these inverse processes can give insight into how biological systems maintain balance and homeostasis.

DNA Transcription and Reverse Transcription

- **DNA Transcription.** The process where DNA is transcribed into RNA:
$$\text{DNA} \to \text{RNA}.$$
- **Reverse Transcription.** An inverse process where RNA is converted back into DNA (occurs in retroviruses, like HIV):
$$\text{RNA} \to \text{DNA}.$$

Photosynthesis and Cellular Respiration

- **Photosynthesis.** Plants convert carbon dioxide and water into glucose and oxygen using sunlight, stores energy in the form of glucose:

$$6CO_2 + 6H_2O + \text{light energy} \rightarrow C_6H_{12}O_6 + 6O_2$$

- **Cellular Respiration.** The inverse process where glucose and oxygen are converted back into carbon dioxide and water, releasing energy:

$$C_6H_{12}O_6 + 6O_2 \rightarrow 6CO_2 + 6H_2O + \text{energy}$$

Formal Definition

A function g is called the **inverse** of a function f if $(f \circ g)(x) = x$ (and automatically $(g \circ f)(x) = x$). The inverse of f is denoted by f^{-1}.

Remark

The inverse of a function is **NOT** the multiplicative inverse, in other words, $f^{-1} \neq \frac{1}{f}$.

Example

Consider $f(x) = 2x - 1$.

- The **multiplicative inverse** of f would be $g(x) = \frac{1}{2x-1}$, since $f(x) \cdot g(x) = (2x - 1) \cdot \frac{1}{2x-1} = 1$.
- However, this $g(x)$ is **not** the inverse function of f: $(f \circ g)(x) = f\left(\frac{1}{2x-1}\right) = 2 \cdot \frac{1}{2x-1} - 1 = \frac{2-(2x-1)}{2x-1} = \frac{3-2x}{2x-1} \neq x$.

So, $g(x) = \frac{1}{2x-1}$ is **not** the inverse of $f(x) = 2x - 1$.

Examples of Inverse Functions

- **Exponential and Logarithmic Functions:**

$$e^x \quad \text{and} \quad \ell n(x);$$
$$a^x \quad (a > 0, a \neq 1) \quad \text{and} \quad \ell og_a(x).$$

- **Trigonometric Functions (restricted domains):**

$$\sin(x) \quad \text{and} \quad \arcsin(x);$$
$$\cos(x) \quad \text{and} \quad \arccos(x);$$
$$\tan(x) \quad \text{and} \quad \arctan(x).$$

Remark

For a function to have an inverse, it must be **one-to-one** (each output corresponds to exactly one input). This is why trigonometric functions must be restricted to specific intervals.

Derivative of an Inverse Function

Recall that $f^{-1}(x)$ is an inverse function of $f(x)$ if $f(f^{-1}(x)) = x$. Taking derivatives with respect to x and applying the chain rule, we get:

$$f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1.$$

Solving for $(f^{-1})'(x)$ gives the general formula:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Example

Let's apply this formula to compute the derivative of the natural logarithm function. Since $\ell n(x)$ is the inverse of e^x and $(e^x)' = e^x$:

$$(\ell n(x))' = \frac{1}{e^{\ell n(x)}} = \frac{1}{x}.$$

Derivatives of $\arcsin(x)$ and $\arccos(x)$

We apply the formula

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

to find the derivative of $\arcsin(x)$.
Since $\arcsin(x)$ is the inverse of $\sin(x)$ and $(\sin(x)' = \cos(x))$, we get:

$$(\arcsin(x))' = \frac{1}{\cos(\arcsin(x))}.$$

Use the identity $\cos(\arcsin(x)) = \sqrt{1 - \sin^2(\arcsin(x))} = \sqrt{1 - x^2}$, so:

$$(\arcsin(x))' = \frac{1}{\sqrt{1 - x^2}}.$$

Similarly, since $\arccos(x)$ is the inverse of $\cos(x)$ and $(\cos(x))' = -\sin(x)$, we get:

$$(\arccos(x))' = \frac{1}{-\sin(\arccos(x))} = -\frac{1}{\sqrt{1 - x^2}}.$$

Derivative of $\arctan(x)$

Since $\arctan(x)$ is the inverse of $\tan(x)$ and $(\tan(x))' = \sec^2(x)$, we get:

$$(\arctan(x))' = \frac{1}{\sec^2(\arctan(x))}.$$

Using the identity $\sec^2(\arctan(x)) = 1 + \tan^2(\arctan(x)) = 1 + x^2$, allows to simplify this expression:

$$(\arctan(x))' = \frac{1}{1 + x^2}.$$

Example

Example

Let $f(x) = 6x^3 + 4$. Use the fact that $f(2) = 52$ to find $(f^{-1})'(52)$.

We begin by applying the inverse function derivative formula:

$$(f^{-1})'(52) = \frac{1}{f'\big(f^{-1}(52)\big)}.$$

To evaluate this, we first compute $f^{-1}(52)$. Since $f(2) = 52$, it follows that $f^{-1}(52) = 2$.. Next, we compute the derivative $f'(x) = 18x^2$, and substitute $x = 2$:

$$f'(2) = 18 \cdot 2^2 = 72.$$

Therefore,

$$(f^{-1})'(52) = \frac{1}{72}.$$

Exercise 13.2

Exercise

Let $f(x) = 5 + 2e^x \sin(x)$. Use the fact that $f(0) = 5$ to find $(f^{-1})'(5)$. Enter your answer as a decimal.

Solution

We are given $f(x) = 5 + 2e^x \sin(x)$, and $f(0) = 5$. We want to compute $(f^{-1})'(5)$.

Using the formula for the derivative of the inverse:

$$(f^{-1})'(5) = \frac{1}{f'\big(f^{-1}(5)\big)}.$$

Since $f(0) = 5$, it follows that $f^{-1}(5) = 0$. Now compute the derivative of $f(x)$ using the product rule:

$$f'(x) = (5+2e^x \sin(x))' = 2e^x \sin(x)+2e^x \cos(x) = 2e^x(\sin(x)+\cos(x)).$$

Therefore,

$$f'(0) = 2e^0(\sin(0) + \cos(0)) = 2 \cdot (0 + 1) = 2,$$

and so

$$(f^{-1})'(5) = \frac{1}{f'\big(f^{-1}(5)\big)} = \frac{1}{f'(0)} = \frac{1}{2}.$$