MATH 007 A

Lecture 12

Derivatives of Trigonometric, Exponential and Logarithmic Functions

This Week's Assignments

- **Homework 4.7:** Extended until *Wednesday* 11/05, 11:59 PM.
- **Homework 4.8, 4.9:** Due on *Monday* 11/10, 11:59 PM.
- **Microtutorial 5:** Due on *Monday* 11/10, 11:59 PM.

Outline

Question 7, HW 4.7: Hill's Equation

Question. The fraction of hemoglobin $f(P)$ that is bound to oxygen depends on the oxygen concentration P, modeled by Hill's equation:

$$f(P) = \frac{P^m}{k^m + P^m},$$

where $k > 0$, $m > 0$ are constants. Show that $f''(0) > 0$ when $m = 2$.

Solution. Substitute $m = 2$ into the formula:

$$f(P) = \frac{P^2}{k^2 + P^2}.$$

Differentiate the first time:

$$f'(P) = \frac{2P \cdot (k^2 + P^2) - P^2 \cdot 2P}{(k^2 + P^2)^2} = \frac{2P(k^2 + P^2) - 2P^3}{(k^2 + P^2)^2} = \frac{2Pk^2}{(k^2 + P^2)^2}.$$

Question 7, HW 4.7: Hill's Equation

Now differentiate again:

$$f''(P) = \left(\frac{2Pk^2}{(k^2+P^2)^2}\right)'$$

Use quotient rule:

$$f''(P) = \frac{2k^2(k^2+P^2)^2 - 2Pk^2 \cdot 2(k^2+P^2) \cdot 2P}{(k^2+P^2)^4}.$$

Evaluate at $P = 0$:

$$f''(0) = \frac{2k^2 \cdot (k^2)^2 - 0}{(k^2)^4} = \frac{2k^6}{k^8} = \frac{2}{k^2} > 0$$

Therefore, $f''(0) > 0$ when $m = 2$.

Question 7, HW 4.7: Hill's Equation

Next we show that $f''(0) < 0$ when $m = 1$.
Substitute $m = 1$ to get the equation $f(P) = \frac{P}{k+P}$
Differentiate the first time:

$$f'(P) = \frac{1 \cdot (k + P) - P \cdot 1}{(k + P)^2} = \frac{k}{(k + P)^2}$$

Differentiate again:

$$f''(P) = \left(\frac{k}{(k + P)^2}\right)' = \left(k(k + P)^{-2}\right)' = k\left((k + P)^{-2}\right)' = -2k(k+P)^{-3}$$

Evaluate at $P = 0$:

$$f''(0) = \frac{-2k}{k^3} = \frac{-2}{k^2} < 0.$$

Therefore, $f''(0) < 0$ when $m = 1$.

Exercise 12.0

Exercise

Find the smallest positive integer n such that the nth derivative of the function

$$f(x) = 17x^4 + 20x^3 - 5x^2 + x - 9$$

is identically zero.

Solution

Each time we differentiate a polynomial, its degree decreases by one. Therefore, once the order of differentiation exceeds the degree of the polynomial, the result becomes identically zero.
Since $\deg f(x) = 4$, the smallest such n is $4 + 1 = 5$.
We can confirm this by computing the derivatives explicitly:

$$f'(x) = 68x^3 + 60x^2 - 10x + 1$$
$$f''(x) = 204x^2 + 120x - 10$$
$$f'''(x) = 408x + 120$$
$$f^{(4)}(x) = 408$$
$$f^{(5)}(x) = 0.$$

Overview

In this lecture, we review the derivatives of basic trigonometric and exponential functions, and learn how to differentiate more complex expressions involving them using the chain rule, product rule, and quotient rule.

Basic Trigonometric Derivatives

The following table summarizes the derivatives of the six basic trigonometric functions:

$f(x)$	$f'(x)$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\cot(x)$	$-\csc^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\csc(x)\cot(x)$

Examples

Example

Compute the following derivatives:

1. $f(x) = \sin(5x - x^2)$.

   We apply the chain rule:

$$f'(x) = \cos(5x - x^2) \cdot (5x - x^2)' = \cos(5x - x^2) \cdot (5 - 2x).$$

2. $g(x) = \frac{1}{\tan^4(\sqrt{x})}$.

   We apply the chain rule:

$$g'(x) = -4\tan^{-5}(\sqrt{x})(\tan(\sqrt{x}))' = -4\tan^{-5}(\sqrt{x})\sec^2(\sqrt{x})(\sqrt{x})' =$$

$$= -4\tan^{-5}(\sqrt{x}) \cdot \sec^2(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}.$$

Examples

Example

Compute the derivative of $h(x) = \dfrac{\sin(x^5)}{\sin^5(x)}$.

We use the chain and quotient rules:

$$u(x) = \sin(x^5), \quad v(x) = \sin^5(x)$$
$$u'(x) = \cos(x^5) \cdot (x^5)' = \cos(x^5) \cdot 5x^4$$
$$v'(x) = 5\sin^4(x) \cdot (\sin(x))' = 5\sin^4(x) \cdot \cos(x)$$
$$h'(x) = \frac{\cos(x^5) \cdot 5x^4 \cdot \sin^5(x) - 5\sin^4(x) \cdot \cos(x) \cdot \sin(x^5)}{\sin^{10}(x)}$$

Exercise 12.1

Exercise

Let $f(x) = \cos(-x^2)$. Compute:

1. $f(\sqrt{\pi})$;
2. $f'(\sqrt{\pi})$;
3. $f''\left(\sqrt{\frac{\pi}{2}}\right)$.

Solution

Let $f(x) = \cos(-x^2)$. Then we compute:

1. $f(\sqrt{\pi}) = \cos(-\pi) = -1$;
2. $f'(x) = (\cos(-x^2))' = -\sin(-x^2) \cdot (-2x) = 2x\sin(-x^2)$, so
$$f'\left(\sqrt{\pi}\right) = 2\sqrt{\pi} \cdot \sin(-\pi) = 0;$$
3. $f''(x) = \left(2x\sin(-x^2)\right)' = 2\sin(-x^2) + 2x \cdot \cos(-x^2) \cdot (-2x) = 2\sin(-x^2) - 4x^2 \cdot \cos(x^2)$, so
$$f''\left(\sqrt{\frac{\pi}{2}}\right) = 2\sin\left(-\frac{\pi}{2}\right) - 4 \cdot \frac{\pi}{2} \cdot \cos\left(\frac{\pi}{2}\right) = -2.$$

Basic Derivatives of Exponential and Logarithmic

The following table summarizes the derivatives of exponential functions:

$f(x)$	$f'(x)$
e^x	e^x
a^x	$\ell n(a)a^x$
$\ell n(x)$	$1/x$
$\ell og_a(x)$	$1/(\ell n(a)x)$

Examples

Example

Compute the following derivatives:

1. $f(x) = e^{x^2+x^{-3}}$. We apply the chain and power rules:

$$f'(x) = e^{x^2+x^{-3}} \cdot \left(x^2 + x^{-3}\right)' = e^{x^2+x^{-3}} \cdot \left(2x - 3x^{-4}\right).$$

2. $g(x) = 3^x x^3$. We apply the product rule (with $u(x) = 3^x$ and $v(x) = x^3$):

$$g'(x) = (3^x)' \cdot x^3 + 3^x \cdot (x^3)' = 3^x \ell n(3) \cdot x^3 + 3^x \cdot 3x^2.$$

3. $h(x) = \ell n(\ell n(x - x^3))$. We apply the chain rule:
$h'(x) = \dfrac{1}{\ell n(x-x^3)} \cdot (\ell n(x - x^3))' = \dfrac{1}{\ell n(x-x^3)} \cdot \dfrac{1}{x-x^3} \cdot (x - x^3)'$
$= \dfrac{1}{\ell n(x-x^3)} \cdot \dfrac{1}{x-x^3} \cdot (1 - 3x^2).$

Examples

Example

Compute the derivative of $f(x) = x^x$.

We observe that x^x is neither a power function x^n nor an exponential function a^x, so standard rules do not directly apply.

We use the identity:

$$x^x = e^{x\ell n(x)}$$

by noting that $\ell n(x)$ and e^x are inverse functions.

Now apply the chain (and product) rule(s):

$$f'(x) = \left(e^{x\ell n(x)}\right)' = e^{x\ell n(x)} \cdot (x\ell n(x))' = x^x \cdot (x \cdot \frac{1}{x} + 1 \cdot \ell n(x)) =$$

$$= x^x(1 + \ell n(x)).$$

Exercise 12.2

Exercise

Let $f(x) = \ell n \left(e^{\sqrt[5]{x}+x^2} \right)$ and compute $f'(1)$, and enter your answer as a decimal.

Solution

We notice that $\ell n(x)$ and e^x are inverse functions, hence

$$f(x) = \ell n\left(e^{\sqrt[5]{x}+x^2}\right) = \sqrt[5]{x} + x^2 = x^{1/5} + x^2.$$

Now compute the derivative:

$$f'(x) = \frac{1}{5}x^{-4/5} + 2x.$$

Evaluating at $x = 1$ gives:

$$f'(1) = \frac{1}{5} \cdot 1^{-4/5} + 2 \cdot 1 = \frac{1}{5} + 2 = \frac{11}{5} = 2.2.$$