MATH 007 A

Lecture 11

Higher Order Derivatives and Their Properties

This Week's Assignments

- **Homework 4.7:** Extended until *Wednesday* 11/05, 11:59 PM.
- **Homework 4.8, 4.9:** Due on *Monday* 11/10, 11:59 PM.
- **Microtutorial 5:** Due on *Monday* 11/10, 11:59 PM.

Outline

Motivation: Beyond First Derivatives

The first derivative $f'(x)$ tells us whether a function is increasing or decreasing near a point.

But this alone isn't enough to describe the full behavior of a function. For example:

- Is the graph curving upwards or downwards?
- Are we at a local maximum or minimum?

To answer these questions, we study *higher-order derivatives*: the second derivative, third derivative, and so on.

Definition: Higher Order Derivatives

The second derivative is simply the derivative of the first derivative:

$$f''(x) = \frac{d^2\left(f'(x)\right)}{d^2x} = \frac{d\left(f'(x)\right)}{dx}.$$

Likewise, we define the third derivative: $f'''(x) = \frac{d}{dx}(f''(x))$ and so on...

Since we can't keep writing too many apostrophes, we use the notation:

$$f^{(n)}(x) = \frac{d^n\left(f(x)\right)}{d^nx}$$

to mean the n-th derivative of f evaluated at x.

Example: Computing Higher Order Derivatives

Example

Let $f(x) = -2x^4$. Then we compute:

1. $f'(x) = \frac{d}{dx}(-2x^4) = -8x^3$;
2. $f''(x) = \frac{d}{dx}(-8x^3) = -24x^2$
3. $f'''(x) = \frac{d}{dx}(-24x^2) = -48x$;
4. $f^{(4)}(x) = \frac{d}{dx}(-48x) = -48$;
5. $f^{(5)}(x) = \frac{d}{dx}(-48) = 0$;
6. we conclude that all higher derivatives are zero:

$$f^{(n)}(x) = 0 \quad \text{for all } n \geq 4.$$

The Factorial and Powers of x

The *factorial* of a positive integer n is defined as:

$$n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n.$$

Example

- $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 6 \cdot 4 = 24;$
- $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 24 \cdot 5 = 120.$

This concept appears naturally when computing the derivatives of powers of x:

$$\frac{d}{dx}(x^n) = nx^{n-1},$$

$$\frac{d^2}{dx^2}(x^n) = \frac{d}{dx}(nx^{n-1}) = n(n-1)x^{n-2},$$

$$\ldots$$

$$\frac{d^n}{dx^n}(x^n) = n!$$

Combinatorial Meaning of Factorial

Remark

The number $n!$ counts the number of different ways to arrange n distinct objects in a sequence.

For example, there are $3! = 6$ ways to order the letters A, B, and C:

1. ABC
2. ACB;
3. BAC;
4. BCA;
5. CAB;
6. CBA.

Exercise 11.1

Exercise

Let $f(x) = \frac{1}{x}$. Compute the following:

1. $f(1)$;
2. $f'(1)$;
3. $f''(1)$;
4. $f'''(1)$;
5. A general formula for $f^{(n)}(1)$ when n is even;
6. A general formula for $f^{(n)}(1)$ when n is odd.

Solution

Let $f(x) = x^{-1}$. We will compute derivatives and evaluate them at $x = 1$:

$$\begin{aligned} f(x) &= x^{-1} \Rightarrow f(1) = 1, \\ f'(x) &= -x^{-2} \Rightarrow f'(1) = -1, \\ f''(x) &= 2x^{-3} \Rightarrow f''(1) = 2 = 2!, \\ f'''(x) &= -6x^{-4} \Rightarrow f'''(1) = -6 = -3!. \end{aligned}$$

The general pattern is:

$$f^{(n)}(x) = (-1)^n \cdot n! \cdot x^{-n-1} \Rightarrow f^{(n)}(1) = (-1)^n \cdot n!$$

Therefore:

- for even n: $f^{(n)}(1) = n!$;
- for odd n: $f^{(n)}(1) = -n!$.

The Role of the Second Derivative

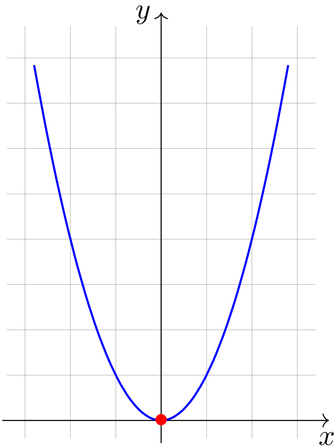
The second derivative tells us about the concavity of the function:

- If $f''(a) > 0$, the graph is *concave up* at a (shaped like a cup);
- If $f''(x) < 0$, the graph is *concave down* at x (shaped like a frown).

In addition, if $f'(a) = 0$ and

- $f''(a) > 0$, then f has a *local minimum* at $x = a$.
- $f''(a) < 0$, then f has a *local maximum* at $x = a$.

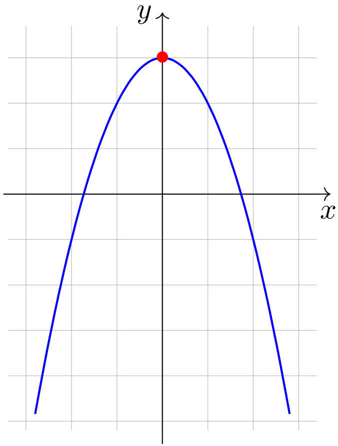
Example: $f(x) = x^2$



- First derivative: $f'(x) = 2x$
- Second derivative: $f''(x) = 2$

Since $f''(x) = 2 > 0$, the graph is concave up everywhere.
Also, $f'(0) = 0$, and $f''(0) = 2 > 0$, so the point $(0, 0)$ is a local (and in this case global) minimum.

Example: $f(x) = 3 - 2x^2$



- First derivative: $f'(x) = -4x$
- Second derivative: $f''(x) = -4$

Since $f''(x) = -4 < 0$, the graph is concave down everywhere.
Also, $f'(0) = 0$, and $f''(0) = -4 < 0$, so the point $(0, 3)$ is a local (and in this case global) maximum.

Second Derivative and Concavity Graphically