MATH 007 A
Lecture 10
Implicit Functions and Implicit Differentiation

This Week's Assignments

- **Homework 4.5:** Extended until *Thursday* 10/30, 11:59 PM.
- **Homework 4.6, 4.7:** Due on *Monday* 11/3, 11:59 PM.
- **Microtutorial 4:** Due on *Monday* 11/3, 11:59 PM.
- **Quiz 3:** complete during your group's Discussion session.

Outline

Chain Rule: One More Example

Find the derivative of $g(t) = \sqrt{7t - \sqrt{6-t}}$.

Step 1. Rewrite in exponent form: $g(t) = \left(7t - (6-t)^{1/2}\right)^{1/2}$.
Step 2. Use the chain (and power) rule(s):

$$g'(t) = \frac{1}{2}\left(7t - \sqrt{6-t}\right)^{-1/2} \cdot \left(7t - (6-t)^{1/2}\right)'.$$

Step 3. Differentiate the inside function:

$$\left(7t - (6-t)^{1/2}\right)' = 7 - \frac{1}{2}(6-t)^{-1/2} \cdot (6-t)' =$$
$$7 - \frac{1}{2}(6-t)^{-1/2} \cdot (-1) = 7 + \frac{1}{2}(6-t)^{-1/2}.$$

Step 4. Plug the result from Step 3 into the expression in Step 2:

$$g'(t) = \frac{1}{2}\left(7t - \sqrt{6-t}\right)^{-1/2}\left(7 + \frac{1}{2}(6-t)^{-1/2}\right).$$

Why Implicit Functions/Differentiation?

Sometimes, it is difficult-or impossible-to solve an equation explicitly for y as a function of x:

- $y^2 - x^2 = 1$: a hyperbola.
- $\frac{x^2}{25} + \frac{y^2}{9} = 1$: an ellipse.
- $\sin(xy) + y^3 = x^2$.

In such cases, we use **implicit differentiation** to compute derivatives like $\frac{dy}{dx}$.

Remark

As we will see, the first two examples (the hyperbola and the ellipse) are not too difficult to solve explicitly, but even there, implicit differentiation provides a quicker method for computing the derivative. In the third example, solving explicitly for y is practically impossible-making implicit differentiation essential.

What is Implicit Differentiation?

If an equation relates x and y, and y is (implicitly) a function of x, then to find the derivative $\frac{dy}{dx}$:

1. Differentiate both sides of the equation with respect to x, using the chain rule on terms involving y.
2. After differentiating, solve the resulting equation for $\frac{dy}{dx}$.

Implicit Differentiation Example: an Ellipse

Example

Consider the equation of an ellipse:

$$\frac{x^2}{25} + \frac{y^2}{9} = 1 \qquad (\star)$$

Differentiate both sides of $(\star)$ with respect to x gives

$$\frac{d}{dx}\left(\frac{x^2}{25} + \frac{y^2}{9}\right) = \frac{d}{dx}(1) \quad \Leftrightarrow \quad \frac{2x}{25} + \frac{2y}{9} \cdot \frac{dy}{dx} = 0$$

Solve for $\frac{dy}{dx}$:

$$\frac{2y}{9} \cdot \frac{dy}{dx} = -\frac{2x}{25} \quad \Leftrightarrow \quad \frac{dy}{dx} = -\frac{9x}{25y}.$$

Slopes of Tangent Lines at Some Points

Example

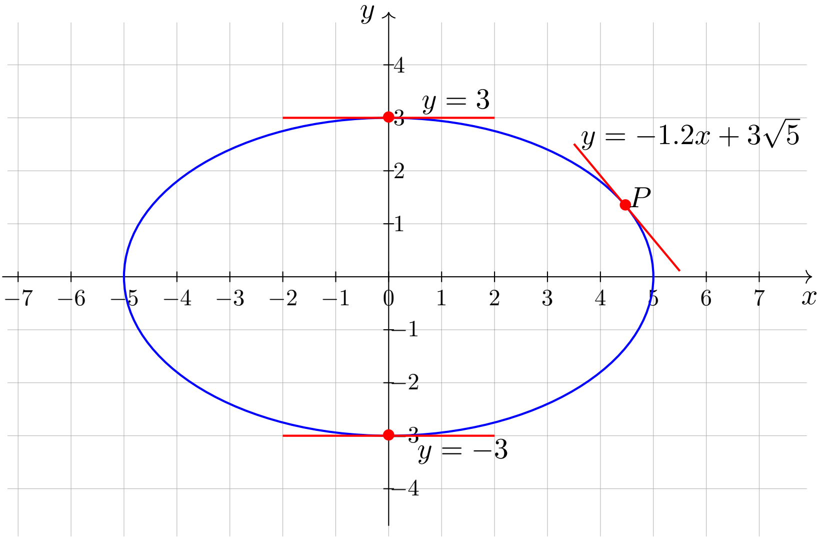
Let's evaluate $\frac{dy}{dx}$ at three points on the ellipse:

- At $(0,3)$: $\frac{dy}{dx} = -\frac{9\cdot 0}{25\cdot 3} = 0$.
- At $(0,-3)$: $\frac{dy}{dx} = -\frac{9\cdot 0}{25\cdot(-3)} = 0$.
- At $P = \left(2\sqrt{5}, \frac{3}{\sqrt{5}}\right)$: $\frac{dy}{dx} = -\frac{9\cdot 2\sqrt{5}}{25\cdot \frac{3}{\sqrt{5}}} = -\frac{18\cdot 5}{75} = -1.2$.

The tangent line at point P has equation

$$y - \frac{3}{\sqrt{5}} = -1.2(x - 2\sqrt{5}) \Rightarrow y = -1.2x + 3\sqrt{5}$$

Visualization: Ellipse and Tangents



Finding the Derivative Explicitly

In the previous example, we could also express y in terms of x explicitly:

$$\frac{x^2}{25} + \frac{y^2}{9} = 1 \Rightarrow y = \pm\sqrt{9 - \frac{9x^2}{25}}.$$

Remark

For a fixed value of x, this equation gives two possible values of y, corresponding to the upper and lower halves of the ellipse. However, if we focus only on one half-say the top half-then $y = \sqrt{9 - \frac{9x^2}{25}}$ defines a function. Similarly, the bottom half corresponds to $y = -\sqrt{9 - \frac{9x^2}{25}}$. Since the notion of the derivative is *local* (i.e., it depends only on behavior of the function near a point), we can compute it on either branch separately.

Finding the Derivative Explicitly

Now let's compute the derivative explicitly using the chain rule:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{9 - \frac{9x^2}{25}}\right) = \frac{1}{2\sqrt{9 - \frac{9x^2}{25}}} \cdot \left(9 - \frac{9x^2}{25}\right)' =$$

$$\frac{1}{2\sqrt{9 - \frac{9x^2}{25}}} \cdot \left(\frac{-18x}{25}\right) = \frac{-9x}{25\sqrt{9 - \frac{9x^2}{25}}} = -\frac{9x}{25y}.$$

Same result! But often, solving for y explicitly is not practical or even possible-especially for more complicated curves.

Implicit Differentiation: Another Example

Consider the equation: $\sin(xy) + y^3 = x^2$. We will compute $\frac{dy}{dx}$ using implicit differentiation.

Fact. The derivative of $\sin(u)$ is $\sin(u)' = \cos(u)$.

Differentiating both sides of with respect to x:

$$\frac{d}{dx}(\sin(xy) + y^3) = \frac{d}{dx}(x^2) \Leftrightarrow \cos(xy) \cdot \frac{d}{dx}(xy) + 3y^2 \cdot \frac{dy}{dx} = 2x.$$

Now apply the product rule to compute $\frac{d}{dx}(xy) = x \cdot \frac{dy}{dx} + y$, giving:

$$\cos(xy)(x \cdot \frac{dy}{dx} + y) + 3y^2 \cdot \frac{dy}{dx} = 2x.$$

Group all $\frac{dy}{dx}$ terms: $\left(\cos(xy) \cdot x + 3y^2\right) \cdot \frac{dy}{dx} = 2x - \cos(xy) \cdot y$. Finally, solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{2x - y\cos(xy)}{x\cos(xy) + 3y^2}.$$

Exercise 10.1

Exercise

Use implicit differentiation to compute $\frac{dy}{dx}$ for the equation

$$2y^3 + 4x^2 - y = 5x^6.$$

Then, determine the slope of the tangent line to the curve at the point $(1,1)$. Enter your answer as a decimal.

Solution

We differentiate both sides of the equation implicitly with respect to x:

$$\frac{d}{dx}(2y^3 + 4x^2 - y) = \frac{d}{dx}(5x^6).$$

Using the chain rule:

$$2 \cdot 3y^2 \cdot \frac{dy}{dx} + 8x - \frac{dy}{dx} = 30x^5 \Leftrightarrow 6y^2 \cdot \frac{dy}{dx} - \frac{dy}{dx} + 8x = 30x^5.$$

Factor out $\frac{dy}{dx}$: $(6y^2 - 1) \cdot \frac{dy}{dx} = 30x^5 - 8x$.
Solve for $\frac{dy}{dx}$: $\frac{dy}{dx} = \frac{30x^5 - 8x}{6y^2 - 1}$.
Evaluate at $(1, 1)$:

$$\frac{30 \cdot 1^5 - 8 \cdot 1}{6 \cdot 1^2 - 1} = \frac{22}{5} = 4.4.$$

So, the slope of the tangent line at $(1, 1)$ is 4.4.

Related Rates Problems

In related rates problems:

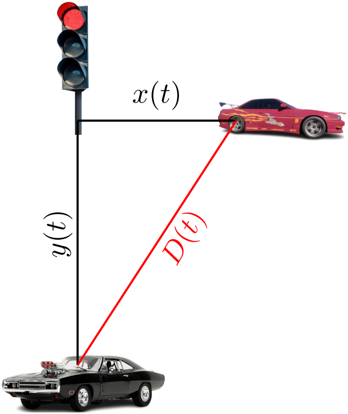
- Two or more quantities are changing with respect to time t.
- We are given the rate of change of one quantity (e.g., $\frac{dx}{dt}$) and asked to find the rate of change of another (e.g., $\frac{dy}{dt}$).
- Use geometry or physical laws to relate the variables, then differentiate **implicitly** with respect to t.

Related Rates: Fast & Furious

In a Fast & Furious chase scene, Dom speeds north from a streetlight at 120 mph, while Letty speeds east from the same point at 90 mph.

How fast is the distance between them increasing after:

- 10 minutes ($\frac{1}{6}$ hr)
- 30 minutes ($\frac{1}{2}$ hr)



Let $x(t) = 90t$ (Letty's eastbound distance), and $y(t) = 120t$ (Dom's northbound distance). We compute:

$$D(t) = \sqrt{x^2(t) + y^2(t)} =$$
$$\sqrt{(90t)^2 + (120t)^2} = \sqrt{22500t^2} = 150t;$$
$$\frac{dD}{dt} = \frac{d}{dt}(150t) = 150\,\text{mph},$$

i.e. the distance between the two cars increases at a constant rate of 150 mph.

Remark

While we could apply the general implicit differentiation formula:

$$\frac{dD}{dt} = \frac{x \cdot dx/dt + y \cdot dy/dt}{\sqrt{x^2 + y^2}},$$

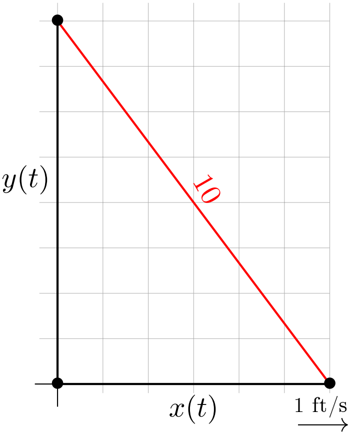
this is not necessary here. Since the distance function simplifies to $D(t) = 150t$, differentiating it directly is much faster and cleaner.

Exercise 10.2

Exercise

A 10 ft ladder is leaning against a vertical wall. The bottom is sliding away at 1 ft/s. How fast is the top sliding down when the bottom is 6 ft from the wall? Enter your answer as a decimal.

Solution



For all times t, the ladder forms a right triangle with hypothenuse of length 10, so

$$x^2(t) + y^2(t) = 10^2 = 100.$$

Differentiating with respect to t gives

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0 \Leftrightarrow x \cdot \frac{dx}{dt} + y \cdot \frac{dy}{dt} = 0.$$

At $x = 6$, we have $y = \sqrt{100 - 36} = 8$. Plugging in $x = 6, y = 8$ and $\frac{dx}{dt} = 1$, we obtain $6 \cdot 1 + 8 \cdot \frac{dy}{dt} = 0 \Leftrightarrow \frac{dy}{dt} = -\frac{6}{8} = -\frac{3}{4} = -0.75$.
So, the top is sliding down at the speed of 0.75 ft/s.