MATH 007 A
Lecture 1
Limits: Getting Acquainted

This Week's Assignments

- **Homework 3.1-3.3:** Due on *Monday* 10/06, 11:59 PM.

Outline

Off Topic: 007 Logo

The number of our class, 007, coincides with the codename of the legendary fictional spy James Bond, created by Ian Fleming. To acknowledge this coincidence, I will use the logo 007 in lecture titles:) If you are unfamiliar with the character, you might enjoy watching some of the classic James Bond films or exploring the original novels. The series is packed with espionage, action, and intrigue-though, unfortunately, none of these will be useful for our coursework!

For an overview of the different James Bond movies, you can check out this website.

The Importance of Limits

The main topic of this lecture will be **limits**. To motivate and demonstrate their importance, we go back to a version of one of the ancient Greek paradoxes, attributed to **Zeno of Elea**.

This paradox challenges our understanding of motion and infinity, and analyzing it rigorously requires the concept of limits.

The Paradox of Achilles and the Tortoise

In a footrace, Achilles is racing a tortoise. He allows the tortoise a 1000 feet head start. Suppose that Achilles runs twice as fast as the tortoise. After a finite time, Achilles will have run 1000 feet, reaching the tortoise's starting point.
During this time, the tortoise has run $\frac{1000}{2} = 500$ feet.
It will then take Achilles more time to cover this 500 feet, by which time the tortoise will have advanced 250 feet farther. Thus, whenever Achilles reaches a point where the tortoise has been, the tortoise is still ahead.

Resolution: The Infinite Series

The situation described above leads to an infinite series:

$$1000 + 500 + 250 + 125 + \dots$$

This is a geometric series where each subsequent term is $q = 0.5$ of the previous one. The sum of an infinite geometric series is given by

$$S = \frac{a}{1-q}$$

where $a = 1000$ is the initial value and $q = 0.5$. Substituting these values, we get

$$S = \frac{1000}{1-0.5} = \frac{1000}{0.5} = 2000.$$

Therefore, Achilles will eventually reach the tortoise after running a total of 2000 feet. Thus, even though Achilles must pass an infinite number of *checkpoints*, he covers only a finite distance of 2000 feet and ultimately catches the tortoise.

Exercise 1.1

Exercise

Which of the following describe Achilles? (Select all that apply)

Achilles in Mythology

Achilles was a demi-god from Greek mythology, son of Thetis (a goddess of water) and Peleus (a mortal king). He was the greatest Greek warrior in the Trojan War, famously described in Homer's *Iliad*. According to legend, his mother dipped him in the River Styx to make him invulnerable. She held him by one of his heels, leaving it untouched by the waters and thus his only vulnerable body part. He was ultimately killed by an arrow to the heel, guided by Paris.
Alluding to these legends, the term Achilles' heel has come to mean a point of weakness.
Achilles has been depicted in various works, including the 2004 movie *Troy*, where he was portrayed by Brad Pitt.

Limits: The Intuitive Idea

The paradox of Achilles and the tortoise is an example of a situation where something seems to involve infinitely many steps, yet has a well-defined outcome. This relates to the mathematical concept of a *limit*.

Intuition: when we say a function $f(x)$ has a limit at some value a, we mean that as x gets *closer and closer* to a, the function values $f(x)$ get *closer and closer* to some fixed number L.

Key Points

- We don't require $x = a$, only that x gets arbitrarily close to a.
- The function might not even be defined at $x = a$.
- The important question is: "What number does $f(x)$ approach?"

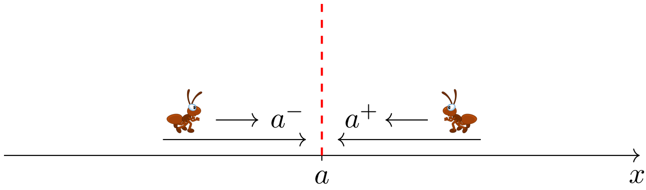
Approaching Particular Values

Question

How does $f(x)$ behave as x *approaches* a particular number a?

To answer this, we first need to clarify the meaning of "approaches":

- from the right, denoted as $x \to a^+$ or
- from the left, denoted as $x \to a^-$



Integers and Rational Numbers, Rational Functions

Recall that the set of numbers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is known as the set of **integers**, encompassing both negative and non-negative whole numbers.
Expanding our number system, we introduce the set of **rational numbers**, defined as

$$\left\{ \frac{a}{b} \mid a, b \text{ are integers and } b \neq 0 \right\}.$$

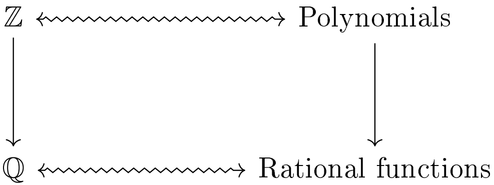
In simpler terms, rational numbers are ratios of integers:
$\{\ldots, \frac{2}{3}, \frac{9}{25}, -\frac{7}{5}, \ldots\}$
Now, we turn our attention to mathematical structures arising from ratios of polynomials.

Definition

Let $P(x)$ and $Q(x) \neq 0$ be two polynomials. We refer to a function $g(x) = \frac{P(x)}{Q(x)}$ as a **rational function**.

Remark

Rational functions are constructed from polynomials in a manner analogous to how rational numbers are constructed from integers. The similarity extends further; for instance, we have polynomial long division, reminiscent of the division process used for integers, illustrating the structural parallels between these mathematical entities.



Domain of Rational Functions

The domain of a rational function $g(x) = \frac{P(x)}{Q(x)}$ consists of all real numbers except for those values that result in a zero denominator (zeros of $Q(x)$).

Example

- The domain of the rational function $f(x) = \frac{75x - 22}{3(5-x)(2x-3)(x+7)}$ is the set $x \neq -7, 1.5, 5$ or, in the interval notation, $(-\infty, -7) \cup (-7, 1.5) \cup (1.5, 5) \cup (5, \infty)$.
- The domain of the rational function $g(x) = \frac{16 - 3x}{(x^2 + 15)(2 - x)}$ is the set $x \neq 2$ or, in the interval notation, $(-\infty, 2) \cup (2, \infty)$.

Vertical asymptotes of a rational function $f(x) = \frac{P(x)}{Q(x)}$ are represented by the vertical lines $x = a$, where a ranges through the zeros of the denominator.

Example

- To find vertical asymptotes of $f(x) = \dfrac{75x - 22}{3(5 - x)(2x - 3)(x + 7)}$, set each factor in the denominator equal to zero:

$$\begin{aligned} 5 - x = 0 &\quad\Longrightarrow\quad x = 5 \\ 2x - 3 = 0 &\quad\Longrightarrow\quad x = 1.5 \\ x + 7 = 0 &\quad\Longrightarrow\quad x = -7 \end{aligned}$$

  The vertical asymptotes are the lines $x = -7, x = 1.5$ and $x = 5$.

- The vertical asymptote of the rational function $g(x) = \dfrac{16 - 3x}{(x^2 + 15)(2 - x)}$ is the line $x = 2$.

Approaching Vertical Asymptotes

Notice that

$$\frac{1}{\text{very small number}} = \text{very large number},$$

for instance,

$$\frac{1}{0.000001} = 1000000,$$

illustrating that $f(x)$ tends towards ∞ or $-\infty$ as $x \to a^+$ or a^-. To determine the sign of $f(x)$ near $x = a$, substitute a in place of x everywhere except in the factor $(x-a)^k$ in the denominator. Observe that:

$$(x-a) > 0 \,\text{if}\, x \to a^+, \text{ and } (x-a) < 0 \,\text{if}\, x \to a^-.$$

This approach helps us analyze the behavior of $f(x)$ near the vertical asymptote $x = a$. By carefully examining the signs of the terms, we can determine whether $f(x)$ approaches positive infinity (∞) or negative infinity ($-\infty$) as x approaches a from the right or left.

Consider the rational function $f(x) = \frac{75x - 22}{3(5-x)(2x-3)(x+7)}$.

For $x \to 5^+$:

$$f(x) \to \frac{75 \cdot 5 - 22}{3(5-x)(2 \cdot 5 - 3)(5+7)} = \frac{353}{3(5-x) \cdot 7 \cdot 12} =$$
$$\frac{353}{252} \cdot \frac{1}{5-x} \to -\infty \quad (\text{as } 5 - x < 0 \text{ for } x > 5).$$

For $x \to 5^-$:

$$f(x) \to \frac{353}{252} \cdot \frac{1}{5-x} \to \infty \quad (\text{as } 5 - x > 0 \text{ for } x < 5).$$

Similarly, for $x \to 1.5^+$:

$$f(x) \to \frac{75 \cdot 1.5 - 22}{3(5-1.5)(2x-3)(1.5+7)} = \frac{90.5}{3 \cdot 3.5 \cdot (2x-3) \cdot 8.5} =$$
$$\frac{90.5}{89.25} \cdot \frac{1}{2x-3} \to \infty \text{ (as } 2x - 3 > 0 \text{ for } x > 1.5).$$

For $x \to 1.5^-$:

$$f(x) \to \frac{90.5}{89.25} \cdot \frac{1}{2x-3} \to -\infty \quad (\text{as } 2x - 3 < 0 \text{ for } x < 1.5).$$

Now, for $x \to -7^+$:

$$f(x) \to \frac{75 \cdot (-7) - 22}{3(5-(-7))(2 \cdot (-7) - 3)(x+7)} = \frac{-547}{3(5+7) \cdot (-17) \cdot (x+7)} = \frac{-571}{-252} \cdot \frac{1}{x+7} = \frac{571}{252} \cdot \frac{1}{x+7} \to \infty \text{ (as } x+7>0 \text{ for } x>-7).$$

Finally, for $x \to -7^-$:

$$f(x) \to \frac{571}{252} \cdot \frac{1}{x+7} \to -\infty \quad (\text{as } x+7<0 \text{ for } x<-7).$$