# Images of skyscraper sheaves on toric resolutions: cohomology distribution

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# Introduction

Let $G \subset SL_n(\mathbb{C})$ be a finite subgroup and consider the affine variety $X = \mathbb{C}^n/G := \operatorname{Spec} \mathbb{C}[x_1, x_2, \ldots, x_n]^G$. We are interested in examples with the following properties

1. $X$ has an isolated singularity at 0;
2. there exists a projective resolution $\pi : Y \to X$
3. there is a bijection

$$\{\text{irr. comp. of } \pi^{-1}(0)\} \xleftrightarrow{1:1} \{\chi \in \operatorname{Irr}(G) \setminus \text{triv}\}$$

A good candidate for such a resolution $Y$ is the $G$-Hilbert scheme $G\text{-Hilb}(\mathbb{C}^n)$.

**Definition.** *A* ***cluster*** $\mathcal{Z} \subset \mathbb{C}^n$ *is a zero-dimensional subscheme and a* **G-*****cluster*** *is a $G$-invariant cluster, s.t. $H^0(\mathcal{O}_\mathcal{Z}) \simeq \mathcal{R}$ (the regular representation of $G$). The* **G-*****Hilbert scheme*** *($G$-Hilb$(\mathbb{C}^n)$) is the fine moduli space parameterizing $G$-clusters.*

# $G$-Hilb$(\mathbb{C})^3$ as a toric variety

Let $G \subset SL_3(\mathbb{C})$ be a finite abelian subgroup of order $r = |G|$, and $\varepsilon = e^{2\pi i/r}$ a primitive root of unity. We diagonalize the action of $G$ and denote the corresponding coordinates on $\mathbb{C}^3$ by $x, y$ and $z$. The lattice of exponents of Laurent monomials in $x, y, z$ will be denoted by $L = \mathbb{Z}^3$ and the dual lattice by $L^\vee$. Next we associate a vector $v_g = \dfrac{1}{r}(\gamma_1, \gamma_2, \gamma_3)$ to a group element $g = \mathrm{diag}(\varepsilon^{\gamma_1}, \varepsilon^{\gamma_2}, \varepsilon^{\gamma_3})$. Write $N := L^\vee + \sum_{g \in G} \mathbb{Z} \cdot v_g$ (with $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$) and $M := \mathrm{Hom}(N, \mathbb{Z})$ for the dual lattice of $G$-invariant Laurent monomials. The categorical quotient $X = \mathrm{Spec}\ \mathbb{C}[x, y, z]^G$ is the toric variety $\mathrm{Spec}\ \mathbb{C}[\sigma^\vee \cap M]$ with the cone $\sigma$ being the positive octant $\sigma = \mathbb{R}_{\geq 0} e_i \subset N_\mathbb{R}$.

**Definition.** *The **junior simplex** $\triangle \subset N_{\mathbb{R}}$ is the subset, which consists of the lattice points* $\frac{1}{r}(\gamma_1, \gamma_2, \gamma_3)$ *inside the standard simplex* $\gamma_1 + \gamma_2 + \gamma_3 = 1, \gamma_i \geq 0$. *The vertices of* $\triangle$ *will be denoted by* $e_x = (1,0,0), e_y = (0,1,0)$ *and* $e_z = (0,0,1)$.

A subdivision of the cone $\sigma$ gives rise to a fan $\Sigma$ and, hence, a toric variety $X_\Sigma$ together with a birational map $X_\Sigma \to X$. A triangle inside the junior simplex is called **basic** in case the pyramid with this triangle as the base and origin as the apex has volume 1. If a fan $\Sigma$ gives rise to a partition of the junior simplex into basic triangles, then the corresponding map $X_\Sigma \to X$ is a crepant resolution of singularities. Notice that such a fan $\Sigma$ is uniquely determined by the associated triangulation of the junior simplex into basic triangles (slightly abusing notation we will refer to such a triangulation as $\Sigma$ as well).

**Fact.** *The $G$-Hilbert scheme is a toric variety and for $G \subset SL_2(\mathbb{C})$ or $SL_3(\mathbb{C})$ the map $Y \to X$ is a crepant resolution of singularities. The partition of the junior simplex into basic triangles, giving rise to the fan of $Y$ and can be computed according to an explicit algorithm.*

# Modern formulation of McKay correspondence

Let $Coh_G(\mathbb{C}^n)$ be the category of $G$-equivariant coherent sheaves on $\mathbb{C}^n$, and $Coh(Y)$ be the category of coherent sheaves on $Y$. The McKay correspondence is the derived equivalence

$$\Psi : D^b(Coh_G(\mathbb{C}^n)) \to D^b(Coh(Y))$$

Any finite-dimensional representation $V$ of $G$ gives rise to two equivariant sheaves on $\mathbb{C}^n$:

- the skyscraper sheaf $V^! = V \otimes_{\mathbb{C}} \mathcal{O}_0$, whose fiber at 0 is $V$ and all the other fibers vanish;
- the locally free sheaf $\widetilde{V} = V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n}$.

# Known results

The McKay correspondence holds in the following cases:

1. $G \subset SL_2(\mathbb{C})$, any $G$ (KV '98)
2. $G \subset SL_3(\mathbb{C})$, any $G$, $Y = G\text{-}Hilb(\mathbb{C}^3)$ (BKR '01)
3. $G \subset SL_3(\mathbb{C})$, any abelian $G$ (CI '04)
4. $G \subset SP_{2n}(\mathbb{C})$, $Y$ is a crepant symplectic resolution (BK '04)
5. $G \subset SL_n(\mathbb{C})$, any abelian $G$, $Y$ is a projective crepant symplectic resolution (Kawamata)

A natural question: what are the images of $\widetilde{\chi}$ and $\chi^!$ $(\chi \in \mathrm{Irr}(G) \setminus triv)$ under the equivalence?

1. $\Psi(\widetilde{\chi})$ is a vector bundle of dimension $\dim(\chi)$ and is called a tautological or GSp-V sheaf (after Gonzales-Sprinberg and Verdier).

2. Relatively little is known about $\Psi(\chi^!)$.

The following results are due to Kapranov, Vasserot and Cautis, Craw, Logvinenko.

**Theorem.** *1. Let $G \subset SL_2(\mathbb{C})$ be a finite subgroup and $\chi \in Irr(G) \setminus triv$. Then $\Psi(\chi^!) \simeq \mathcal{O}_{\mathbb{P}^1_\chi}(-1)[1]$.*

*2. Let $G \subset SL_3(\mathbb{C})$ be a finite abelian subgroup, s. t. $X = \mathbb{C}^3/G$ has an isolated singularity at the origin. Then for any $\chi \in Irr(G) \setminus triv$, the object $\Psi(\chi^!) \in D^b(Coh(Y))$ is pure (here $Y = G\text{-}Hilb(\mathbb{C}^3)$ and an object is called* ***pure*** *provided all cohomology groups, except one, vanish).*

# Reid's recipe

Reid's recipe is an algorithm to construct the cohomological version of the McKay correspondence for abelian subgroups of $SL_3(\mathbb{C})$. It is based on marking **internal** edges and vertices of the triangulation $\Sigma$ corresponding to $G$-Hilb with characters of G.

**Theorem.** *Let $G \subset SL_3(\mathbb{C})$ be a finite abelian subgroup and let $\chi$ be an irreducible representation of $G$. Then $H^i(\Psi(\chi^!)) = 0$ unless $i \in \{0, -1, -2\}$. Moreover, one of the following holds:*

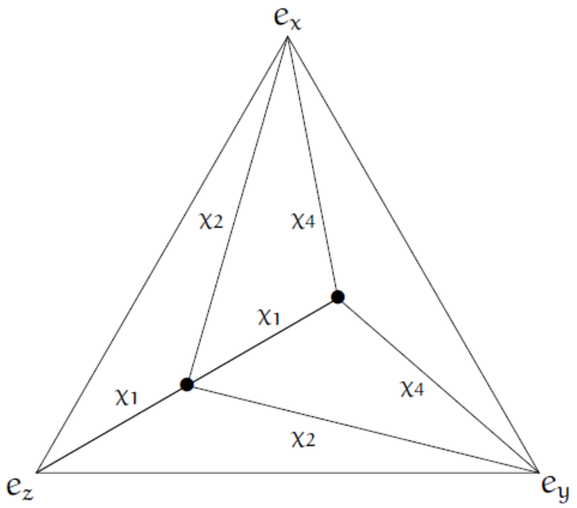
| *Reid's recipe* | $H^{-2}(\Psi(\chi^!))$ | $H^{-1}(\Psi(\chi^!))$ | $H^0(\Psi(\chi^!))$ |
|---|---|---|---|
| $\chi$ *marks a single divisor* $E$ | $0$ | $0$ | $\mathcal{L}_\chi^{-1} \otimes \mathcal{O}_E$ |
| $\chi$ *marks a single curve* $C$ | $0$ | $0$ | $\mathcal{L}_\chi^{-1} \otimes \mathcal{O}_C$ |
| $\chi$ *marks a chain of divisors starting at* $E$ *and terminating at* $F$ | $0$ | $\mathcal{L}_\chi^{-1}(-E-F) \otimes \mathcal{O}_Z$ | $0$ |
| $\chi$ *marks three chains of divisors, starting at* $E_x$, $E_y$ *and* $E_z$ *and meeting at a divisor* $P$ | $0$ | $\mathcal{L}_\chi^{-1}(-E_x - E_y - E_z) \otimes \mathcal{O}_{\mathcal{V}_Z}$ | $0$ |
| $\chi = \chi_0$ | $w_{ZF_2}$ | $w_{ZF_1}(ZF_2)$ | $0$ |

**Question.** *Let* $\mathfrak{H}_0$ *be the set of nontrivial characters* $\chi : G \to \mathbb{C}^*$ *with* $H^0(\Psi(\chi^!)) \neq 0$*. What are the possible values of* $\frac{|\mathfrak{H}_0|}{r-1}$*, the proportion of nontrivial characters* $\chi$ *with* $\Psi(\chi^!)$ *concentrated in degree* $0$*?*

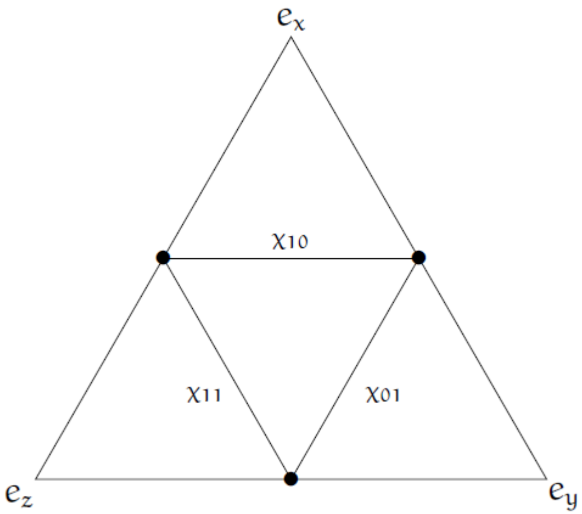
**Theorem.** *One has* $0.25 \leq \frac{|\mathfrak{H}_0|}{r-1} \leq 1$*, both the upper and lower bounds are sharp.*

*Proof.* Each of the triangles in partition of junior simplex $\triangle$ corresponding to $G$-$\mathrm{Hilb}(\mathbb{C}^3)$ is basic, so it is of area $\frac{1}{r}\cdot\mathcal{S}(\triangle)$ with $r=|G|$. As the triangles do not overlap and cover $\triangle$, it follows that there are $r$ triangles in the triangulation. Furthermore, each inside edge (not contained in the boundary of $\triangle$) is adjacent to two basic triangles and the number of such edges does not exceed $\dfrac{3r-3}{2}$ (there are at least 3 edges on the boundary). It follows from the classification on $11^{\mathrm{th}}$ slide that each character $\chi$ with $H^{-1}(\Psi(\chi^!))\neq 0$ marks at least two edges. This gives rise to the lower bound $1-\dfrac{3r-3}{4(r-1)}=\dfrac{1}{4}\leq\dfrac{|\mathfrak{H}_0|}{r-1}$. $\square$

Consider the group $G = \mathbb{Z}/5\mathbb{Z}$ with $\nu_1 = \frac{1}{5}(1,1,3)$, then $\frac{|\mathfrak{H}_0|}{r-1} = 1 - \frac{3}{4} = \frac{1}{4}$.
We conclude with verifying the assertion on the upper bound. Let $\varphi$ : $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \hookrightarrow SL_3(\mathbb{C})$ be given by $\varphi(1,0) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and $\varphi(0,1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, then $\frac{|\mathfrak{H}_0|}{r-1} = 1 - \frac{0}{3} = 1$



$\Sigma$ fan and character labels for $G = \frac{1}{5}(1,1,3)$



$\Sigma$ fan and character labels for $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

# Questionnaire

We conclude with a collection of possible directions for further research.

Let $G$ be a cyclic abelian group with generator $1 \in G$. Then any embedding $G \hookrightarrow SL_3(\mathbb{C})$ is up to conjugation given by $\varphi_{ab} : G \hookrightarrow SL_3(\mathbb{C})$ with $\varphi_{ab} = \mathrm{diag}(\zeta, \zeta^a, \zeta^b)$, where $\zeta = e^{2\pi i/r}$ and $r = |G|$. Let $\mathcal{S} := \{a, b \in \mathbb{Z}^2_{>0} \mid a+b = r-1\}$ and define the function $\mathfrak{G} : \mathcal{S} \to [0.25, 1] \cap \mathbb{Q}$ via $\mathfrak{G}(a, b) := \dfrac{|\mathfrak{H}_0|}{r-1}$ (computed with respect to the embedding $\varphi_{ab}$). Notice that $\mathfrak{G}(a, b) = \mathfrak{G}(b, a)$.

1. What are the max/min of $\mathfrak{G}$?

2. What is the range of $\mathfrak{G}$?

3. What is the distribution of the values in the range of $\mathfrak{G}$?

**Example.** *Consider the groups* $G = \mathbb{Z}/2k\mathbb{Z}$ *and* $\widetilde{G} = \mathbb{Z}/(2k+1)\mathbb{Z}$ *with* $\nu_1 = \frac{1}{2k}(1,1,2k-2)$ *and* $\widetilde{\nu}_1 = \frac{1}{2k+1}(1,1,2k-1)$. *Then* $\frac{|\mathfrak{H}_0|}{2k-1} = \frac{k-1}{2k-1}$ *and* $\frac{|\mathfrak{H}_0|}{2k} = \frac{k-1}{2k}$, *respectively.*