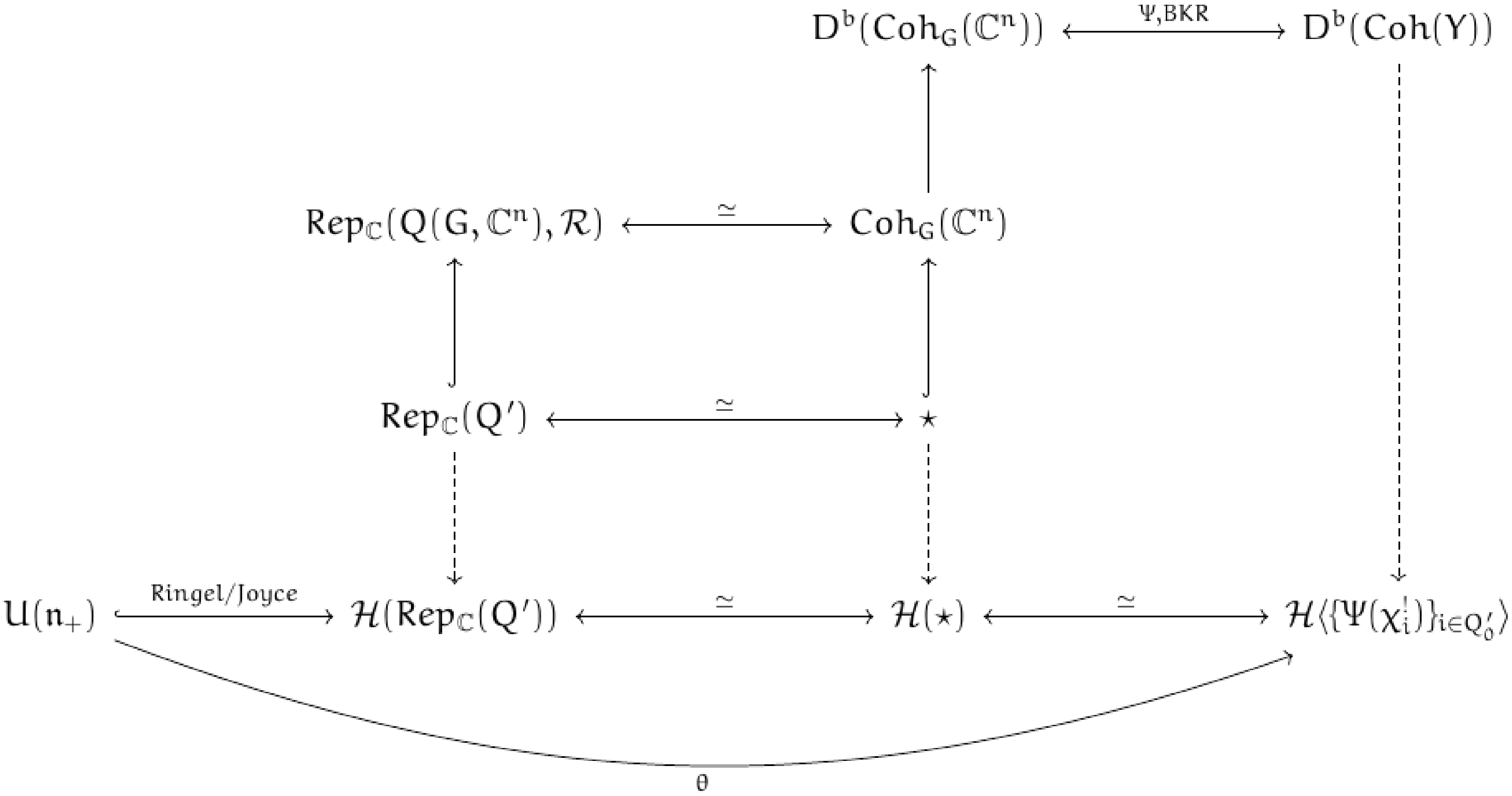
The universe inside Hall algebras of coherent sheaves on toric resolutions

Boris Tsvelikhovskiy

Motivation



$D^b(\mathrm{Coh}_G(\mathbb{C}^n))$
Ψ,BKR
$D^b(\mathrm{Coh}(Y))$
$\mathrm{Rep}_{\mathbb{C}}(Q(G,\mathbb{C}^n),\mathcal{R})$
$\simeq$
$\mathrm{Coh}_G(\mathbb{C}^n)$
$\mathrm{Rep}_{\mathbb{C}}(Q')$
$\simeq$
$\star$
$U(\mathfrak{n}_+)$
Ringel/Joyce
$\mathcal{H}(\mathrm{Rep}_{\mathbb{C}}(Q'))$
$\simeq$
$\mathcal{H}(\star)$
$\simeq$
$\mathcal{H}\langle\{\Psi(\chi_i^!)\}_{i\in Q_0'}\rangle$
θ

22 slides before...

Hall Algebras

Let $\mathcal{C}$ be a small abelian category, such that

- $\mathrm{gldim}(\mathcal{C}) < \infty$, i.e. $\mathrm{Ext}^n(A, B) = 0$ for any $A, B \in Ob(\mathcal{C})$ and $n \gg 0$;
- $|\mathrm{Ext}^i(A, B)| < \infty$ for any $A, B \in Ob(\mathcal{C})$ and all $i \geq 0$.

Definition. *The* ***multiplicative Euler form*** $\langle \cdot, \cdot \rangle : K(\mathcal{C}) \times K(\mathcal{C}) \to \mathbb{C}$ *is the form given by*

$$\langle A, B \rangle := \sqrt{\prod_{i=0}^{\infty} |Ext^i(A, B)|^{(-1)^i}}$$

Let $\mathcal{C}^{iso}$ be the set of isomorphism classes of objects in $\mathcal{C}$ and consider the vector space $\mathcal{H}(\mathcal{C}) := \bigoplus_{A \in \mathcal{C}^{iso}} \mathbb{C}[A]$. The following operation defines the structure of an associative algebra on $\mathcal{H}(\mathcal{C})$:

$$[A] \star [B] := \langle A, B \rangle \sum_{C} P_{A,B}^{C} [C],$$

where $\mathcal{P}_{A,B}^{C}$ is the number of short exact sequences (SES) $0 \to B \to C \to A \to 0$ and $P_{A,B}^{C} := \dfrac{\mathcal{P}_{A,B}^{C}}{|\mathrm{End}(A)||\mathrm{End}(B)|}$.

Remark. *The unit $i : \mathbb{C} \to \mathcal{H}(\mathcal{C})$ is given by $i(\lambda) = \lambda[0]$, where 0 is the initial object of $\mathcal{C}$.*

Example. *Let $\mathcal{C}$ be the category of finite-dimensional vector spaces over a finite field $\Bbbk = \mathbb{F}_q$. The classes of objects in $\mathcal{C}^{iso}$ are given by $\{V_n\}_{n\geq 0}$ with $V_n := \Bbbk^n$. We notice that $Ext^{>0}(V_n, V_m) = 0$, while $|Ext^0(V_n, V_m)| = |Hom(V_n, V_m)| = q^{nm}$. Moreover, the number of SES*

$$0 \to V_n \to V_s \to V_m \to 0$$

is zero, unless $s = m+n$. In case $s = m+n$ the number of SES as above, up to isomorphism of the first and third term, is $|Gr_\Bbbk(n, m+n)|$, where $Gr_\Bbbk(n, m+n)$ is the Grassmannian of n-dimensional subspaces in $(m+n)$-dimensional vector space. We conclude that $[V_m] \star [V_n] = q^{nm/2} \begin{bmatrix} n+m \\ n \end{bmatrix}_q V_{n+m}$, where $\begin{bmatrix} n+m \\ n \end{bmatrix}_q := \frac{[n+m]_q!}{[n]_q![m]_q!}$ with $[n]_q! := \dfrac{(q^n-1)(q^{n-1}-1)\ldots(q-1)}{(q-1)^n}$ is the q-binomial coefficient. It is equal to the number of points on $Gr_\Bbbk(n, m+n)$.

Remark. *Notice that similar to the binomial coefficients, their 'q-analogs' satisfy the equality* $\begin{bmatrix} n+m \\ n \end{bmatrix}_q = \begin{bmatrix} n+m \\ m \end{bmatrix}_q$. *It follows that* $[V_m] \star [V_n] = [V_n] \star [V_m]$ *and the Hall algebra* $\mathcal{H}(\mathcal{C})$ *is commutative. Moreover, the algebra* $\mathcal{H}(\mathcal{C})$ *is generated by* $[V_1]$ *and isomorphic to the ring of polynomials in one variable* $\mathbb{C}[x]$. *It is straightforward to check that* $[V_1]^{\star n} = \sqrt{q^{\frac{n(n-1)}{2}}}[n]_q![V_n]$ *(notice that* $[n]_q!$ *is the number of points on the variety of complete flags over* $\mathbb{F}_q$ *and* $q^{\frac{n(n-1)}{2}} = q^{1\cdot 1}q^{1\cdot 2}\dots q^{1\cdot(n-1)}$*). Hence, the isomorphism of algebras* $\varphi : \mathbb{C}[x] \xrightarrow{\sim} \mathcal{H}(\mathcal{C})$ *with* $\varphi(x) = [V_1]$ *has* $\varphi(x^n) = q^{\frac{n(n-1)}{4}}[n]_q![V_n]$.

Quivers

Definition. *A* ***quiver*** $Q = (Q_0, Q_1)$ *is a finite directed graph with finitely many vertices enumerated by the set* Q_0 *and finitely many edges indexed by* Q_1*. Each edge is uniquely determined by the pair of vertices it connects, which we will denote by* $t(a)$ *and* $h(a)$ *standing for 'tail' and 'head', respectively. A* ***representation of a quiver*** Q *consists of a collection of vector spaces* $\{V_i\}_{i \in Q_0}$ *and linear homomorphisms* $\alpha_a \in Hom(V_{ta}, V_{ha})$ *for each arrow* $a \in Q_1$*.*

Such representations form a category with morphisms being collections of $\mathbb{k}$-linear maps $\psi_i : V_i \to W_i$ for all $i \in Q_0$ such that the diagrams

$$\begin{array}{ccc} V_{ta} & \xrightarrow{\alpha_a} & V_{ha} \\ \downarrow{\scriptstyle \psi_{ta}} & & \downarrow{\scriptstyle \psi_{ha}} \\ W_{ta} & \xrightarrow{\alpha'_a} & W_{ha} \end{array}$$

commute.

This category will be denoted by $\mathrm{Rep}(Q)$. There is a natural way to associate a Kac-Moody Lie algebra $\mathfrak{g}_Q$ to Q. Namely, the Cartan matrix for $\mathfrak{g}_Q$ is $C = 2 \cdot I - A_Q - A_Q^T$, where A_Q is the adjacency matrix of Q.

Definition. *A* ***path*** *p in a quiver $Q = (Q_0, Q_1)$ is a sequence $a_\ell a_{\ell-1} \dots a_1$ of arrows in Q_1 such that $t(a_{i+1}) = h(a_i)$ for $i = 1, 2, \dots, \ell - 1$. In addition, for every vertex $x \in Q_0$ we introduce a path e_x.*

The ***path algebra*** *$\mathcal{P}_Q$ is a $\Bbbk$-algebra with a basis labeled by all paths in Q. The multiplication in $\mathcal{P}_Q$ is given by*

$$p \cdot q := \begin{cases} pq, & \text{if } t(p) = h(q) \\ 0, & \text{otherwise,} \end{cases}$$

where pq stands for the concatenation of paths subject to the conventions that $pe_x = p$ if $t(p) = x$, and $e_x p = p$ if $h(p) = x$.

Remark. *Notice that* $\mathcal{P}_Q$ *is of finite dimension over* $\Bbbk$ *if and only if* Q *has no oriented cycles. The path algebra has a natural grading by path length with the subring of grade zero spanned by the trivial paths* e_x *for* $x \in Q_0$. *It is a semisimple ring, in which the elements* e_x *are orthogonal idempotents.*

Theorem. *The category* $Rep(Q)$ *is equivalent to the category of finitely-generated left* $\mathcal{P}_Q$*-modules. In particular,* $Rep(Q)$ *is an abelian category.*

Remark. *If* Q *has no oriented cycles, then the category* $Rep(Q)$ *is hereditary, i.e.* $Ext^i(A, B) = 0$ *for any* $i \geq 2$ *and* $A, B \in Rep(Q)$.

Example. *Let* Q *be the quiver* $\bullet_1 \longrightarrow \bullet_2$. *An object in* $Rep(Q)$ *is a pair of vector spaces* (V_1, V_2) *together with a linear map* $a \in Hom(V_1, V_2)$. *There are two simple objects* $S_1 : \Bbbk \to 0$ *and* $S_2 : 0 \to \Bbbk$ *and one (up to isomorphism) indecomposable, which is not simple* $I_{12} = \Bbbk \xrightarrow{id} \Bbbk$. *The adjacency matrix* $A_Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *gives rise to Cartan matrix* $C = 2 \cdot I - A_Q - A_Q^T = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ *corresponding to Lie algebra* $\mathfrak{sl}_3$ *of traceless* 3×3 *matrices. The path algebra,* $\mathcal{P}(Q)$, *is of dimension* 3 *over* $\Bbbk$. *It is generated by two idempotents* e_1, e_2 *and an element* a *subject to relations* $ae_1 = e_2 a = a$, $e_1^2 = e_1, e_2^2 = e_2$ *and* $ae_2 = e_1 a = a^2 = 0$.

Remark. *Notice that we have a natural bijection between simple roots* $E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ *in* $\mathfrak{sl}_3$ *and simple objects in* $Rep(Q)$.

Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra with $\mathfrak{g}_Q$ the Lie algebra associated to the Dynkin diagram formed by Q. We denote the simple roots of $\mathfrak{g}_Q$ by $\{E_i\}_{i\in Q_0}$ and simple representations of Q by $\{S_i\}_{i\in Q_0}$.

The following result was obtained by Ringel and Green.

Theorem. *Let $\Bbbk$ be a finite field and $v = \sqrt{|\Bbbk|}$. There is an embedding of algebras $\varphi : U_v(\mathfrak{n}_+) \hookrightarrow \mathcal{H}(Rep_{\Bbbk}(Q))$ with $\varphi(E_i) = [S_i]$ (here $\mathfrak{n}_+$ is the standard maximal nilpotent subalgebra in $\mathfrak{g}_Q$).*

Let $\mathcal{C} = R\text{-}mod$ be a category of finite–dimensional left modules over a fixed finite–dimensional, associative $\mathbb{C}$–algebra R. There is a way to associate a Hall algebra $\mathcal{H}(\mathcal{C})$ to $\mathcal{C}$. The construction was sketched by Kapranov and Vasserot and later given in detail by Joyce. Notice that if Q has no oriented cycles, then its path algebra $\mathcal{P}(Q)$ has the required property, therefore, we can associate a Hall algebra to the category of finite–dimensional left modules over this algebra. The latter is equivalent to $Rep(Q)$.

Theorem. *There is an embedding of algebras $\varphi : U(\mathfrak{n}_+) \hookrightarrow \mathcal{H}(Rep_{\mathbb{C}}(Q))$.*

McKay correspondence

Let $G \subset GL_n(\mathbb{C})$ be a finite subgroup and consider the affine variety $X = \mathbb{C}^n//G := Spec(\mathbb{C}[x_1, x_2, \ldots, x_n])^G$. We are interested in examples with the following properties

1. X has an isolated singularity at 0;
2. there exists a projective resolution $\pi : Y \to X$

Definition. *The bijection*

$$\{irr.\ comp.\ of\ \pi^{-1}(0)\} \xleftrightarrow{1:1} \{\rho \in Irr(G) \setminus triv\}$$

is known as McKay correspondence.

A good candidate for such a resolution Y is the G-Hilbert scheme $G\text{-Hilb}(\mathbb{C}^n)$.

Definition. *A* ***cluster*** $\mathcal{Z} \subset \mathbb{C}^n$ *is a zero-dimensional subscheme and a* **G-*cluster*** *is a* G*-invariant cluster, s.t.* $H^0(\mathcal{O}_{\mathcal{Z}}) \simeq \mathcal{R}$ *(the regular representation of* G*). The* **G-*Hilbert scheme*** $(G\text{-}Hilb(\mathbb{C}^n))$ *is the fine moduli space parameterizing* G*-clusters.*

Example. *Let* $G = \mathbb{Z}_r$ *be embedded into* $SL_2(\mathbb{C})$ *via*

$$\varphi : \mathbb{Z}_r \hookrightarrow SL_2(\mathbb{C}),\ \varphi(1) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \text{ with } \zeta = e^{\frac{2\pi i}{r}}.$$

Then $X = Spec(\mathbb{C}[x,y])^G \simeq \mathbb{C}[u,v,w]/(uv - w^r)$ *with* $u = x^r, v = y^r, w = xy$.

Using the definition of G-Hilb, we get

$$Y := G\text{-Hilb}(\mathbb{C}^2) = \{I_{\mathcal{Z}} \subset \mathbb{C}[x,y] \mid G \cdot I_{\mathcal{Z}} \subseteq I_{\mathcal{Z}}, \mathbb{C}[x,y]/I_{\mathcal{Z}} \simeq \mathcal{R}\},$$

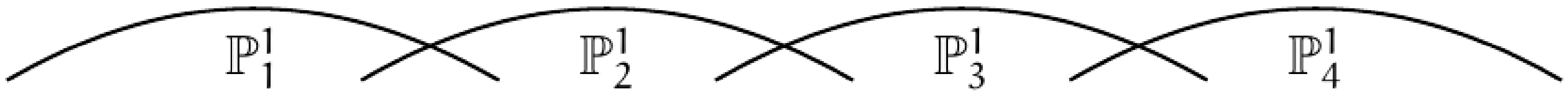
where $\mathcal{R} \simeq \bigoplus_{i=0}^{n} \chi_i$ for $\chi_i : \mathbb{Z}_r \to \mathbb{C}^*$, $\chi_i(1) = \zeta^i$.

Fact. *Y is smooth and the map $\pi : Y \to X$ given by $\pi(I_{\mathcal{Z}}) = supp(I_{\mathcal{Z}})$ is a projective resolution. Moreover, X has an isolated singularity at the origin. The central fiber is*

$$\pi^{-1}(0) = \bigcup_{j=1}^{r-1} \mathcal{I}_j;$$

$$\mathcal{I}_j := \{I_{\lambda_j,\mu_j} = \langle \lambda_j x^j - \mu_j y^{r-j}, xy, x^{j+1} \rangle \mid (\lambda_j, \mu_j) \in \mathbb{C}^2 \backslash (0,0)\} \simeq \mathbb{P}^1 = [\lambda_j : \mu_j].$$

Remark. $\mathcal{I}_j \cap \mathcal{I}_k = \begin{cases} pt, & |k-j| = 1 \\ \varnothing & otherwise. \end{cases}$



$\pi^{-1}(0)$, type A_4 Kleinian singularity

is dual to

Dynkin diagram A_4

McKay Quiver

Definition. • *Let $G \subset GL(V)$ be a finite abelian subgroup. The **McKay quiver** $Q(G,V)$ is the graph given by the following data*

$$\{vertices\ of\ Q\} \xleftrightarrow{1:1} \{irreps\ of\ G\}$$

$$\#\{edges\ i \to j\} = dim(Hom_G(\chi_i \otimes V, \chi_j)).$$

- $Rep_{\mathbb{C}}(Q(G,\mathbb{C}^n),\mathcal{R})$ *is the category of representations of* $Q(G,V)$ *satisfying the conditions:*

$$x_{jk}x_{ij} = x_{kj}x_{ik}$$

where $x_{ij} \in Hom(V_{\chi_i}, V_{\chi_j})$.

Remark. *Representations of G are one-dimensional and correspond to characters of G:*

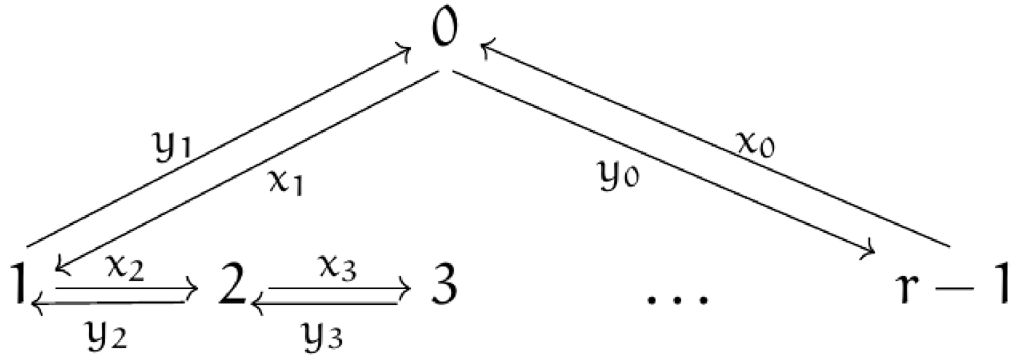
$$char(G) := \{\chi : G \to \mathbb{C}^*\}.$$

In particular, as a representation of G, we have $\mathbb{C}^n = \bigoplus_{i=1}^{n} \mathbb{C}\chi_i =: \bigoplus_{i=1}^{n} \mathbb{C}e_i$ and let $x_1, x_2, \ldots, x_n \in (\mathbb{C}^n)^*$ be the dual basis to $\{e_1, e_2, \ldots, e_n\}$ with $R = \mathbb{C}[x_1, x_2, \ldots, x_n]$ the coordinate ring of $\mathbb{C}^n$. The chain of isomorphisms $Hom_G(\chi_k \otimes \mathbb{C}^n, \chi_\ell) \simeq Hom_G(\chi_k \otimes \bigoplus_{i=1}^{n} \mathbb{C}e_i, \chi_\ell) \simeq \bigoplus_{i=1}^{n} Hom_G(\chi_k \otimes \mathbb{C}e_i, \chi_\ell)$ provides a natural identification of the maps assigned to the arrows in the McKay quiver $Q(G, \mathbb{C}^n)$ with multiplication by x_i's and, hence, imposes the relations corresponding to commutation of the latter that we saw on the previous slide.

Example. *Let $G = \mathbb{Z}_r$ be embedded into $SL_2(\mathbb{C})$ via*

$$\varphi : \mathbb{Z}_r \hookrightarrow GL_2(\mathbb{C}),\ \varphi(1) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \text{ with } \zeta = e^{\frac{2\pi i}{r}}.$$

There are r irreducible representations of G, one-dimensional, to be denoted by $\chi_0, \chi_1, \ldots, \chi_{r-1}$ and $\mathbb{C}^2 \simeq \chi_1 \oplus \chi_{r-1}$. We label $x_k := a^{\chi_k}_{\chi_1}$ and $y_k := a^{\chi_k}_{\chi_{r-1}}$, then the ideal of relations is $\mathcal{I} = \langle x_i y_i - x_{i+1} y_{i+1} \rangle$.



Modern formulation of McKay correspondence

Let $Coh_G(\mathbb{C}^n)$ be the category of G-equivariant coherent sheaves on $\mathbb{C}^n$, and $Coh(Y)$ be the category of coherent sheaves on Y. The McKay correspondence is the derived equivalence

$$\Psi : D^b(Coh_G(\mathbb{C}^n)) \to D^b(Coh(Y)).$$

Any finite-dimensional representation V of G gives rise to two equivariant sheaves on $\mathbb{C}^n$: the skyscraper sheaf $V^! = V \otimes_{\mathbb{C}} \mathcal{O}_0$, whose fiber at 0 is V and all the other fibers vanish, and the locally free sheaf $\widetilde{V} = V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n}$.

Remark. *There is an equivalence of abelian categories*

$$\Theta : Rep(Q(G, \mathbb{C}^n), \mathcal{R}) \simeq Coh_G(\mathbb{C}^n).$$

Known results

The derived McKay correspondence holds in the following cases:

1. $G \subset SL_2(\mathbb{C})$, any G (KV '98)
2. $G \subset SL_3(\mathbb{C})$, any G, $Y = G\text{-}Hilb(\mathbb{C}^3)$ (BKR '01)
3. $G \subset SL_3(\mathbb{C})$, any abelian G (CI '04)
4. $G \subset SP_{2n}(\mathbb{C})$, Y is a crepant symplectic resolution (BK '04)
5. $G \subset SL_n(\mathbb{C})$, any abelian G, Y is a projective crepant symplectic resolution (Kawamata)

A natural question: what are the images of $\widetilde{\chi}$ and $\chi^!$ $(\chi \in \mathrm{Irr}(G) \setminus \mathit{triv})$ under the equivalence?

1. $\Psi(\widetilde{\chi})$ is a vector bundle of dimension $\dim(\chi)$ and is called a tautological or GSp-V sheaf (after Gonzales-Sprinberg and Verdier).

2. Relatively little is known about $\Psi(\chi^!)$.

The following results are due to Kapranov, Vasserot and Logvinenko.

Theorem. *1. Let $G \subset SL_2(\mathbb{C})$ be a finite subgroup and $\chi \in Irr(G) \setminus triv$. Then $\Psi(\chi^!) \simeq \mathcal{O}_{\mathbb{P}^1_\rho}(-1)[1]$.*

2. Let $G \subset SL_3(\mathbb{C})$ be a finite abelian subgroup, s.t. $X = \mathbb{C}^3//G$ has an isolated singularity at the origin. Then for any $\chi \in Irr(G) \setminus triv$, the object $\Psi(\chi^!) \in D^b(Coh(Y))$ is pure (here $Y = G\text{-}Hilb(\mathbb{C}^3)$ and an object is called ***pure*** *provided all cohomology groups, except one, vanish).*

Remark. *The KV result gives a natural way to associate nontrivial irreps with irreducible components of the central fiber (this is consistent with the correspondence that we established earlier).*

Suppose $G \subset SL_3(\mathbb{C})$ satisfies the following assumptions:

1. The McKay quiver $Q(G, \mathbb{C}^3)$ contains a subquiver Q' (without oriented cycles) with $\mathcal{R} \cap \mathcal{P}_{Q'} = 0$;
2. Ψ sends the skyscraper sheaves $\chi^! \in Coh_G(\mathbb{C}^n)$, corresponding to the simple representations in $\mathrm{Rep}(Q(G, \mathbb{C}^3), \mathcal{R})$ supported at the vertices of Q', to pure sheaves concentrated in the same degree.

Kapranov and Vasserot have also observed that if $\mathcal{C}_1$ and $\mathcal{C}_2$ are $\mathbb{C}$-linear finitary abelian categories, there is a derived equivalence $\Psi : D^b(\mathcal{C}_1) \to D^b(\mathcal{C}_2)$ and a collection of objects $\{a_1, \ldots, a_n\}$ in $\mathcal{C}_1$, s.t. $\Psi(a_i)$ are all pure and concentrated in the same degree, then the Hall algebra generated by the objects $\{a_1, \ldots, a_n\}$ is isomorphic to the Hall algebra generated by their images $\{\Psi(a_1), \ldots, \Psi(a_n)\}$.

Let $\mathcal{H}\langle\{\Psi(\chi_i^!)\}_{i\in Q_0'}\rangle$ be the Hall algebra generated by the images of sheaves corresponding to simple representations of Q' under Ψ and $\mathfrak{n}_+ \subset \mathfrak{g}_{Q'}$ stand for the corresponding nilpotent subalgebra of $\mathfrak{g}_{Q'}$. It follows from the discussion above that one has an isomorphism of algebras:

$$\theta : U(\mathfrak{n}_+) \to \mathcal{H}\langle\{\Psi(\chi_i^!)\}_{i\in Q_0'}\rangle.$$

In [2] I have found an infinite collection of cyclic finite abelian subgroups of $SL_3(\mathbb{C})$ satisfying the aforementioned conditions for each simply laced Dynkin diagram Q' of affine type except $\widetilde{D}_4$. If you would like to learn more about Hall algebras, [1] is an excellent place to start!

References

[1] O. Schiffmann, *Lectures on Hall algebras*, Geometric methods in representation theory. II, Sémin. Congr., vol. 24, Soc. Math. France, Paris, 2012, pp. 1–141 (English, with English and French summaries).

[2] B. Tsvelikhovskiy, *The universe inside Hall algebras of coherent sheaves on toric resolutions*, arXiv:2201.07847v2 (2022).

The diagram revisited