On G-Hilb scheme and McKay correspondence

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Introduction

Let $G \subset GL_n(\mathbb{C})$ be a finite subgroup and consider the affine variety $X = \mathbb{C}^n/G := Spec(\mathbb{C}[x_1, x_2, \ldots, x_n])^G$. We are interested in examples with the following properties

1. X has an isolated singularity at 0;
2. there exists a projective resolution $\pi : Y \to X$
3. there is a bijection

$$\{\text{irr. comp. of } \pi^{-1}(0)\} \xleftrightarrow{1:1} \{\rho \in \mathrm{Irr}(G) \setminus \mathrm{triv}\}$$

A good candidate for such a resolution Y is the G-Hilbert scheme $G\text{-Hilb}(\mathbb{C}^n)$.

Definition. *A* ***cluster*** $\mathcal{Z} \subset \mathbb{C}^n$ *is a zero-dimensional subscheme and a* **G-cluster** *is a G-invariant cluster, s.t. $H^0(\mathcal{O}_{\mathcal{Z}}) \simeq \mathcal{R}$ (the regular representation of G). The* **G-*Hilbert scheme*** (G-Hilb($\mathbb{C}^n$)) *is the fine moduli space parameterizing G-clusters.*

Example. *Let $G = \mathbb{Z}_r$ be embedded into $SL_2(\mathbb{C})$ via*

$$\varphi : \mathbb{Z}_r \hookrightarrow SL_2(\mathbb{C}),\ \varphi(1) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \text{ with } \zeta = e^{\frac{2\pi i}{r}}.$$

Then $X = Spec(\mathbb{C}[x,y])^G \simeq \mathbb{C}[u,v,w]/(uv - w^r)$ with $u = x^r, y = v^r, w = xy$.

Using the definition of G-Hilb, we get

$$Y := G - \mathrm{Hilb}(\mathbb{C}^2) = \{I_{\mathcal{Z}} \subset \mathbb{C}[x,y] \mid H^0(\mathcal{O}(\mathcal{Z})) = \mathbb{C}[x,y]/I_{\mathcal{Z}} \simeq \mathcal{R}\},$$

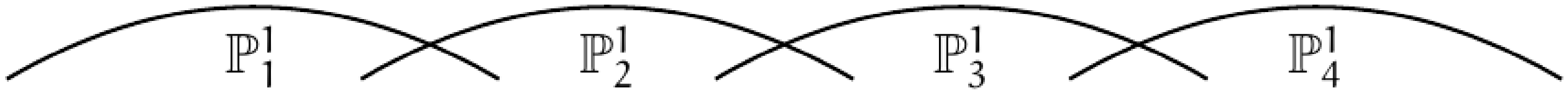
where $\mathcal{R} \simeq \bigoplus_{i=0}^{n} \rho_i$ for $\rho_i : \mathbb{Z}_r \to \mathbb{C}^*$, $\rho_i(1) = \zeta^i$.

Fact. *Y is smooth and the map $\pi : Y \to X$ given by $\pi(I_{\mathcal{Z}}) = supp(I_{\mathcal{Z}})$ is a projective resolution. Moreover, X has an isolated singularity at the origin. The central fiber is*

$$\pi^{-1}(0) = \bigcup_{j=1}^{r-1} I_{\lambda_j,\mu_j}$$

with $I_{\lambda_j,\mu_j} = \langle \lambda_j x^j - \mu_j y^{r-j}, xy, x^{j+1} \rangle \simeq \mathbb{P}^1 = [\lambda_j : \mu_j]$.

Remark. $I_{\lambda_j,\mu_j} \cap I_{\lambda_k,\mu_k} = \begin{cases} pt, & |k-j| = 1 \\ \varnothing & otherwise. \end{cases}$



$\pi^{-1}(0)$, type A_4 Kleinian singularity

dual

Dynkin diagram A_4

Fact. *Let $G \subset SL_2(\mathbb{C})$ be a finite subgroup, $\pi : G-Hilb(\mathbb{C}^2) \to X$ the crepant projective resolution. Then*

1. *the number of irreducible components (E_i) of $\pi^{-1}(0)$ coincides with the number of nontrivial irreps of G;*

2. *the graph, dual to the intersection graph of E_i's is the Dynkin diagram (subgraph of McKay quiver $Q = (G, V)$), in particular, the Cartan matrix is the negative of the intersection matrix (with entries $E_{ij} := E_i \cdot E_j$):*

Example. *Type A*

$$C_n = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix} \text{ and } E_n = \begin{pmatrix} -2 & 1 & 0 & 0 & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & -2 & 1 \\ 0 & 0 & \dots & 0 & 1 & -2 \end{pmatrix}$$

McKay Quiver

Definition. *Let $G \subset GL(V)$ be a finite abelian subgroup. The* ***McKay quiver*** *$Q(G, V)$ is the graph given by the following*

$$\{vertices\ of\ Q\} \overset{1:1}{\longleftrightarrow} \{irreps\ of\ G\}$$

$$\#\{edges\ i \to j\} = dim(Hom_G(\rho_i \otimes V, \rho_j))$$

A ***representation*** *of Q is an additional collection of data: assign a vector space V_ρ of dimension $dim(\rho) = 1$ to every vertex (according to the irrep ρ it is associated to) and a linear map (number) $x_{ij} \in Hom(V_{\rho_i}, V_{\rho_j})$ to every edge $i \to j$ subject to the relations*

$$\langle x_{jk} x_{ij} = x_{kj} x_{ik} \rangle$$

Example. *Again, set* $Q = (\mathbb{Z}_r, \mathbb{C}^2)$ *and let* $\mathcal{I} \subset \mathbb{C}[y_i, x_j]$ *be the ideal, generated by relations* $\langle y_i x_i - x_{i+1} y_{i+1} \rangle$*, set* $R := \mathbb{C}[x_j, y_i]/\mathcal{I}$*, then* $\mathcal{Z} := Spec(R)$ *is the affine scheme formed by representations of* Q *with dimension vector* $(1, 1, \ldots, 1)$*. Next we would like to consider a certain moduli space of such representations. Let* $\mathcal{G} := PGL_G(R) = P\left(\prod_{k=0}^{r} GL_G(\rho_k)\right) = P(\mathbb{C}^*)^r (\simeq (\mathbb{C}^*)^{r-1})$ *be the group of* G*-equivariant automorphisms of* R*. In particular,* $\mathcal{G}$ *acts on* R *via*

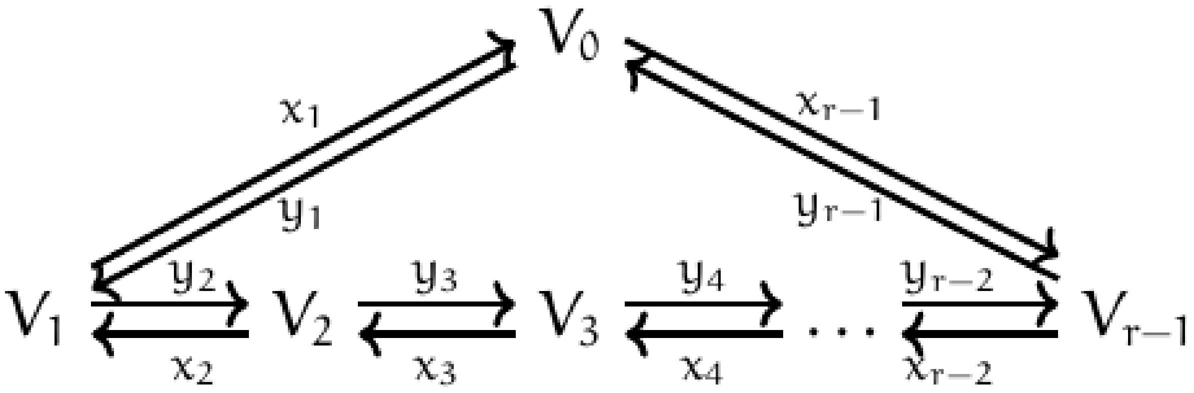
$$(t_1 \ldots t_n) \cdot (x_1, \ldots, x_r, y_1, \ldots, y_r) = (t_2 x_1 t_1^{-1}, \ldots, t_1 x_r t_r^{-1}, t_1 y_1 t_2^{-1}, \ldots, t_r y_r t_1^{-1}).$$

Consider the character $\theta : \mathcal{G} \to \mathbb{C}^*$ given by $\theta(t_1 \dots t_{r-1}) = t_1^{-1} \dots t_{r-1}^{-1}$ and form the categorical quotient $\mathcal{M}_Q^\theta := \mathcal{Z}//_\theta \mathcal{G} = Proj(\bigoplus_{i>0} \mathcal{R}^{\theta,i})$, where

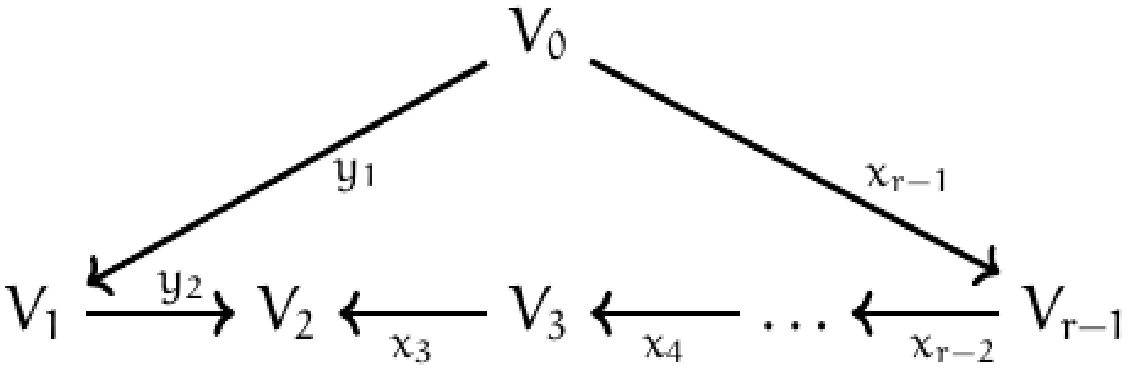
$$\mathcal{R}^{\theta,i} = \{f \in \mathcal{R} \mid f(g^{-1}z) = \theta(g)^i f(z), \forall g \in \mathcal{G}, z \in \mathcal{Z}\}.$$

Define $\mathcal{Z}^{\theta-ss} := \{r \in R \mid \exists k \in \mathbb{N}, f \in \mathcal{R}^{\theta,k}, \text{ s.t. } f(r) \neq 0\}$. Two points $p_1, p_2 \in \mathcal{Z}^{\theta-ss}$ are equivalent ($p_1 \sim_\mathcal{G} p_2$) if the closures of their $\mathcal{G}$-orbits intersect. Notice (with the help of two Davids: Hilbert and Mumford) that for our choice of θ, $z = (x_j, y_i) \in \mathcal{Z}^{\theta-ss}$ if and only if any nonzero vector $v_0 \in V_0$ is cyclic, i.e. the images of v_0 under y_i's and x_j's span all V_k's. There is a bijection between the set of equivalence classes of G-orbits in $\mathcal{Z}^{\theta-ss}$ and the set of geometric points of the scheme $\mathcal{M}_Q^\theta$.

Theorem. *There is an isomorphism* $G - Hilb(\mathbb{C}^2) \simeq \mathcal{M}_Q^{\theta}$.



McKay quiver $Q = (G, \mathbb{C}^2)$



Divisor $E_2 \simeq \mathbb{P}^1 = [x_{r-1}x_{r-2}\dots x_3 : y_1y_2]$

Modern formulation of McKay correspondence

Let $Coh_G(\mathbb{C}^n)$ be the category of G-equivariant coherent sheaves on $\mathbb{C}^n$, and $Coh(Y)$ be the category of coherent sheaves on Y. The McKay correspondence is the derived equivalence

$$\Psi : D^b(Coh_G(\mathbb{C}^n)) \to D^b(Coh(Y))$$

Any finite-dimensional representation V of G gives rise to two equivariant sheaves on $\mathbb{C}^n$: the skyscraper sheaf $V^0 = V \otimes_{\mathbb{C}} \mathcal{O}_0$, whose fiber at 0 is V and all the other fibers vanish, and the locally free sheaf $\widetilde{V} = V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n}$.

Remark. *There is an equivalence of abelian categories*

$$\Theta : Rep(Q, V) \to Coh_G(\mathbb{C}^n).$$

Known results

The McKay correspondence holds in the following cases:

1. $G \subset SL_2(\mathbb{C})$, any G (KV '98)
2. $G \subset SL_3(\mathbb{C})$, any G, $Y = G - Hilb(\mathbb{C}^3)$ (BKR '01)
3. $G \subset SL_3(\mathbb{C})$, any abelian G (CI '04)
4. $G \subset SP_{2n}(\mathbb{C})$, Y is a crepant symplectic resolution (BK '04)
5. $G \subset SL_n(\mathbb{C})$, any abelian G, Y is a projective crepant symplectic resolution (Kawamata)

A natural question: what are the images of $\widetilde{\rho}$ and ρ^0 ($\rho \in \mathrm{Irr}(G)\setminus triv$) under the equivalence?

1. $\Psi(\widetilde{\rho})$ is a vector bundle of dimension $\dim(\rho)$ and is called a tautological or GSp-V sheaf (after Gonzales-Sprinberg and Verdier).

2. Relatively little is known about $\Psi(\rho^0)$.

The following results are due to Kapranov, Vasserot and Logvinenko.

Theorem. *1. Let $G \subset SL_2(\mathbb{C})$ be a finite subgroup and $\rho \in Irr(G) \setminus triv$. Then $\Psi(\rho^0) \simeq \mathcal{O}_{\mathbb{P}^1_\rho}(-1)[1]$.*

*2. Let $G \subset SL_3(\mathbb{C})$ be a finite abelian subgroup, s. t. $X = \mathbb{C}^3/G$ has an isolated singularity at the origin. Then for any $\rho \in Irr(G) \setminus triv$, the object $\Psi(\rho^0) \in D^b(Coh(Y))$ is pure (here $Y = G - Hilb(\mathbb{C}^3)$ and an object is called **pure** provided all cohomology groups, except one, vanish).*

Remark. *The KV result gives a natural way to associate nontrivial irreps with irreducible components of the central fiber (this is consistent with the correspondence that we established last time).*

Ongoing project

(jt. with T. Logvinenko)

Let $G \subset GL_n(\mathbb{C})$ be a cyclic subgroup of order r; choose and fix a generator $g \in G$. Henceforth we make the following assumptions:

1. $g = diag(\xi^{a_1}, \ldots, \xi^{a_n})$ with each $a_i \in \{1, r-1\}$;
2. set $s := \#\{i | a_i = 1\}$, then $0 < s < n$.

Proposition. *1. The variety $Y = G - Hilb(\mathbb{C}^n)$ is smooth.*

2. There are $r-1$ irreducible components in the central fiber $\rho^{-1}(0)$.

3. There is a natural way to associate irreducible component E_ρ to a non-trivial irrep ρ via $supp(H^0(\Psi(\rho^0)))$.

Remark. *As the resolution is not crepant, there is no equivalence between* $D_G(\mathbb{C}^n)$ *and* $D^b(Coh(Y))$*, however, there is a fully faithful functor and the image is a block. The images of the skyscraper sheaves coming from irreps of* G *are never pure.*